Lesson 9: Inversion and Composition of Rigid Motions

Next we begin an exploration of the *algebraic* properties of rigid motions. Every rigid motion can be *inverted* to obtain a rigid motion that reverses the effect of the first rigid motion. Also two rigid motions can be combined by first performing one of the motions and then performing the other. This combination is a new rigid motion called the *composition* of the original two rigid motions. In this lesson we will study what happens when we invert and compose rigid motions. Our study will reveal surprising relationships between different rigid motions. These relationships form the skeleton of a structure that organizes the collection of all rigid motions into a more understandable group of mathematical objects.

We begin by introducing notation for the simplest possible rigid motion.

Definition. Let Π be a plane. Consider the "motion" of the plane Π that keeps every point of Π fixed. In other words, consider the "motion" of Π in which no point moves. We call this motion the *identity motion* of Π (because every point retains its identity), and we denote it either by I_Π, or simply by I if there is no possibility of confusion about which plane Π is. Thus, for each point A of Π,

$$I_\Pi(A) = I(A) = A.$$

The identity motion is a rigid motion because since I_Π doesn't move points, it can't change the distance between them. Indeed, if A and B are points of Π, then $I_\Pi(A) = A$ and $I_\Pi(B) = B$. So $(I_\Pi(A))(I_\Pi(B)) = AB$.

Observe that the identity motion of a plane Π can be thought of as a translation of Π that moves a point to itself. In other words, if A is any point of the plane Π, the $I_\Pi = T_{A,A}$. Also observe that the identity motion of Π can be thought of as a rotation of Π around a point C through an oriented angle of measure 0. In other words, if C is any point of Π, then $I_\Pi = R_{C,0}$.

Next we explore the process of inverting a rigid motion.

Definition. Suppose that M is a rigid motion of a plane Π. Then there is another rigid motion which reverses the moves that M makes. Thus, if A is a point of Π and M moves A to the point B, then this second rigid motion moves B back to A. We call this second rigid motion the *inverse* of M, and we denote it by M^{-1}. Hence, for points A and B of Π:

$$\text{if } M(A) = B, \text{ then } M^{-1}(B) = A.$$

Now suppose A and B are points of Π and $M^{-1}(A) = B$. Because of the way M^{-1} is defined, this can only happen if $M(B) = A$. We have just proved that for points A and B of Π:

$$\text{if } M^{-1}(A) = B, \text{ then } M(B) = A.$$

In other words, from the fact that M^{-1} reverses the motion of M, we have proved that M reverses the motion of M^{-1}. Hence, from the fact that M^{-1} is the inverse of M, we have proved that M is the inverse of M^{-1}. We can express this fact symbolically by writing

$$(M^{-1})^{-1} = M.$$

The *Classification Theorem for Rigid Motions of a Plane* tells us that every rigid motion of a plane is either a translation, a rotation, a reflection or a glide reflection. Since the inverse of a translation is a rigid motion, then it follows that the inverse of a translation must be either a translation, a rotation, a reflection or a glide reflection. The same is true for the inverses of rotations, reflections and glide reflections. In other words, the inverse of each rigid motion must be a rigid motion of one of the four types. We now investigate the inverses of the four types of rigid motions to discover their type.

Activity 1. The class as a whole should discuss and solve the following five problems.

a) Suppose $T_{A,B}$ is a translation of a plane Π. Then the inverse $(T_{A,B})^{-1}$ of $T_{A,B}$ is also a rigid motion of Π. What type of rigid motion is $(T_{A,B})^{-1}$? Draw a convincing picture to support your answer. Furthermore, express $(T_{A,B})^{-1}$ in terms of our notation for rigid motions. In other words, fill in the blank in the following equation with the correct notation:

$$(T_{A,B})^{-1} = \underline{\hspace{4cm}}.$$

b) Suppose $R_{C,a}$ is a rotation of a plane Π. Then the inverse $(R_{C,a})^{-1}$ of $R_{C,a}$ is also a rigid motion of Π. What type of rigid motion is $(R_{C,a})^{-1}$? Draw a convincing picture to support your answer. Furthermore, express $(R_{C,a})^{-1}$ in terms of our notation for rigid motions. In other words, fill in the blank in the following equation with the correct notation:

$$(R_{C,a})^{-1} = \underline{\hspace{4cm}}.$$

c) Suppose Z_L is a reflection of a plane Π. Then the inverse $(Z_L)^{-1}$ of Z_L is also a rigid motion of Π. What type of rigid motion is $(Z_L)^{-1}$? Draw a convincing picture to support your answer. Furthermore, express $(Z_L)^{-1}$ in terms of our notation for rigid motions. In other words, fill in the blank in the following equation with the correct notation:

$$(Z_L)^{-1} = \underline{\hspace{4cm}}.$$

d) Suppose $G_{A,B}$ is a glide reflection of a plane $\prod$. Then the inverse $(G_{A,B})^{-1}$ of $G_{A,B}$ is also a rigid motion of $\prod$. What type of rigid motion is $(G_{A,B})^{-1}$? Draw a convincing picture to support your answer. Furthermore, express $(G_{A,B})^{-1}$ in terms of our notation for rigid motions. In other words, fill in the blank in the following equation with the correct notation:

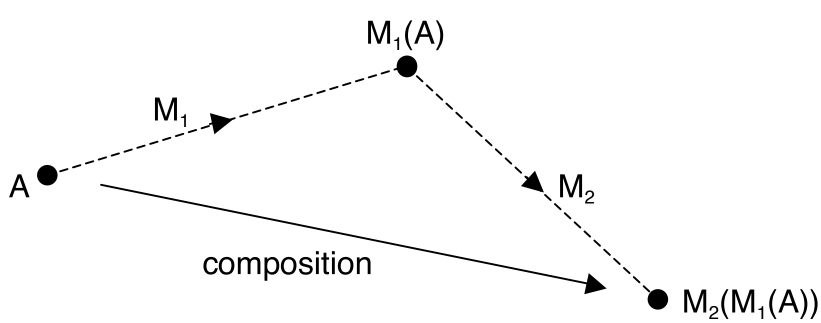
$$(G_{A,B})^{-1} = \underline{\hspace{4cm}}.$$

e) Suppose $I_{\prod}$ is the identity motion of the plane $\prod$. Then the inverse $(I_{\prod})^{-1}$ of $I_{\prod}$ is also a rigid motion of $\prod$. What type of rigid motion is $(I_{\prod})^{-1}$? Draw a convincing picture to support your answer. Furthermore, express $(I_{\prod})^{-1}$ in terms of our notation for rigid motions.. In other words, fill in the blank in the following equation with the correct notation:

$$(I_{\prod})^{-1} = \underline{\hspace{4cm}}.$$

We now study the process of *composing* two rigid motions.

Definition. Suppose that M_1 and M_2 are rigid motions of the plane $\prod$. Consider the motion of $\prod$ produced by first performing the rigid motion M_1 and then performing the rigid motion M_2. Under this two-step motion, a point A of $\prod$ is first moved by M_1 to the point $M_1(A)$. Then the point $M_1(A)$ is moved by M_2 to the point $M_2(M_1(A))$. So the combined effect of this two-step process (first performing M_1 then performing M_2) moves each point A of $\prod$ to the point $M_2(M_1(A))$. We call this two-step motion the *composition of M_1 and M_2*.



We have already seen an example of a composition of two rigid motions: the glide reflection $G_{A,B}$ is the composition of the translation $T_{A,B}$ and the reflection $Z_{\overleftrightarrow{AB}}$.

Again suppose that M_1 and M_2 are rigid motions of the plane $\prod$. Since both M_1 and M_2 preserve distance, then their composition must also preserve distance. For assume A and B are points of the plane Π. Since M_1 preserves distance, then the distance from A to B equals the distance from $M_1(A)$ to $M_1(B)$; in other words,

$$AB = (M_1(A))(M_1(B)).$$

Also, since M_2 preserves distance, then the distance from $M_1(A)$ to $M_1(B)$ equals the distance from $M_2(M_1(A))$ to $M_2(M_1(B))$; in other words,

$$(M_1(A))(M_1(B)) = (M_2(M_1(A)))(M_2(M_1(B)))$$

Putting these two equations together, we get:

$$AB = (M_2(M_1(A)))(M_2(M_1(B))).$$

Thus, the distance from A to B equals the distance from $M_2(M_1(A))$ to $M_2(M_1(B))$. Therefore, the composition of M_1 and M_2 preserves distance. Hence, the composition of M_1 and M_2 is a rigid motion. We have proved that:

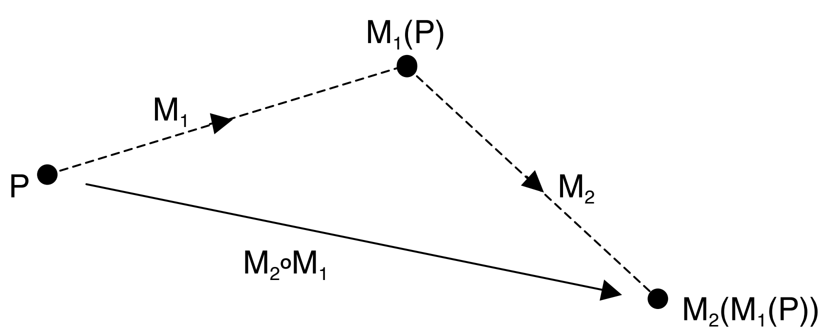
the composition of two rigid motions is again a rigid motion.

Notation. Suppose that M_1 and M_2 are rigid motions of the plane Π. Then (as we just observed) their composition is a rigid motion. We will denote this rigid motion by

$$M_2 \circ M_1.$$

Thus, $M_2 \circ M_1$ is our symbol for the composition of M_1 and M_2. Hence, for any point P of Π,

$$M_2 \circ M_1(P) = M_2(M_1(P)).$$



We may express the fact that the glide reflection $G_{A,B}$ is the composition of the translation $T_{A,B}$ and the reflection $Z_{\overleftrightarrow{AB}}$ in terms of this notation, by writing:

$$G_{A,B} = Z_{\overleftrightarrow{AB}} \circ T_{A,B}.$$

Composition is a sort of multiplication operation for the rigid motions of a plane ∏. We can form the composition of any two rigid motions of ∏ to get a new rigid motion of ∏. We can compose two rigid motions of the same type: two translations, two rotations, two reflections, two glide reflections. Also we can compose rigid motions of different types: a translation and a rotation, a translation and a reflection (as we did to obtain a glide reflection), a rotation and a reflection, etc. In fact, there are $4 \times 4 = 16$ different possible ways to form compositions of the four types of rigid motions. We list these 16 possibilities in the following table. The *Classification Theorem for Rigid*

$T_{A,B} \circ T_{C,D}$	$R_{C,a} \circ T_{A,B}$	$Z_L \circ T_{A,B}$	$G_{A,B} \circ T_{C.D}$
$T_{A,B} \circ R_{C,a}$	$R_{C,a} \circ R_{D,b}$	$Z_L \circ R_{C,a}$	$G_{A.B} \circ R_{C,a}$
$T_{A,B} \circ Z_L$	$R_{C,a} \circ Z_L$	$Z_L \circ Z_M$	$G_{A.B} \circ Z_L$
$T_{A,B} \circ G_{C,D}$	$R_{C,a} \circ G_{A,B}$	$Z_L \circ G_{A,B}$	$G_{A,B} \circ G_{C,D}$

Motions of a Plane tells us that every one of the 16 compositions listed in this table must be one of the four types of rigid motions – a translation, a rotation, a reflection or a glide reflection. In subsequent activities we will explore which types of rigid motions result from each of the 16 compositions in the table. We will begin this exploration in Activity 2 with a very important case. We will investigate the types of rigid motions that can result from the composition of *reflections*.

Before beginning our exploration of the types of rigid motions that result from the composition of two rigid motions, we make two simple observations that relate the concepts of identity, inverse and composition.

Observation 1: Let M be a rigid motion of a plane ∏. Then for each point A of ∏,

$$I_{\Pi} \circ M(A) = I_{\Pi}(M(A)) = M(A) \qquad \text{and} \qquad M \circ I_{\Pi}(A) = M(I_{\Pi}(A)) = M(A).$$

Thus, the three functions $I_{\Pi} \circ M$, $M \circ I_{\Pi}$ and M have the same effect on each point A of the plane ∏. In other words, the three functions $I_{\Pi} \circ M$, $M \circ I_{\Pi}$ and M are identical:

$$I_{\Pi} \circ M = M = M \circ I_{\Pi}.$$

Another way to express this fact is: *Among rigid motions of the plane ∏, I_{Π} is the identity element with respect to the operation of composition.*

Observation 2: Again let M be a rigid motion of a plane ∏. Let A be any point of ∏. We know that if B is a point of ∏ such that M(A) = B, then $M^{-1}(B) = A$. Hence,

$$M^{-1} \circ M(A) = M^{-1}(M(A)) = M^{-1}(B) = A = I_{\Pi}(A).$$

Similarly, we know that if C is any point of Π such that $M^{-1}(A) = C$, then $M(C) = A$. Hence,

$$M \circ M^{-1}(A) \ = \ M(M^{-1}(A)) \ = \ M(C) \ = \ A \ = \ I_{\Pi}(A).$$

Thus, the three functions $M^{-1} \circ M$, $M \circ M^{-1}$ and I_{Π} have the same effect on each point A of the plane Π. In other words, the three functions $M^{-1} \circ M$, $M \circ M^{-1}$ and I_{Π} are identical:

$$M^{-1} \circ M \ = \ I_{\Pi} \ = \ M \circ M^{-1}.$$

Another way to express this fact is: *Among rigid motions of the plane Π, M^{-1} is the inverse of the element M with respect to the operation of composition.*

As you may have noticed, there is a strong analogy between the composition of rigid motions of a plane Π and the multiplication of positive real numbers. We list some of the specific connections that create this analogy.

- The composition of two rigid motions of the plane Π is again a rigid motion of Π just as the product of two positive real numbers is again a real number.
- The identity motion I_{Π} behaves much like the positive real number 1. The composition of I_{Π} with any rigid motion M yields M: $I_{\Pi} \circ M \ = \ M \circ I_{\Pi} \ = \ M$. Similarly the product of 1 with any positive real number a yields a: $1 \times a \ = \ a \times 1 \ = \ a$.
- The inverse M^{-1} of a rigid motion M behaves like the multiplicative inverse $a^{-1} = 1/a$ of a real number a. The composition of M^{-1} and M yields the identity I_{Π}: $M^{-1} \circ M \ = \ M \circ M^{-1} \ = \ I_{\Pi}$. Similarly the product of a^{-1} and a yields 1: $a^{-1} \times a \ = \ a \times a^{-1} \ = \ 1$.

The composition of rigid motions and the multiplication of positive real numbers are both *operations on a set* that share the three properties mentioned above. Mathematicians call a set that is equipped with an operation and has these properties a *group.* Thus, the rigid motions of a plane Π form a group with respect to the operation of composition, and the positive real numbers form a group with respect to the operation of multiplication. Finally we mention one huge difference between the group of rigid motions of a plane Π and the group of positive real numbers.

- Multiplication of real number is *commutative:* in other words, the product of two positive real numbers doesn't depend on the order of the numbers in the product: $a \times b \ = \ b \times a$ for all positive real numbers a and b. However, in general, composition of rigid motions is *not* commutative: if M_1 and M_2 are rigid motions of a plane Π, then usually $M_1 \circ M_2 \ \neq \ M_2 \circ M_1$. (In other words, $M_1 \circ M_2 \ \neq \ M_2 \circ M_1$ except if M_1 and M_2 are related in some special way.) If $M_1 \circ M_2 \ = \ M_2 \circ M_1$, then we say that M_1 and M_2 *commute.* Thus, in general, two rigid motions don't commute.

The properties of rigid motions that we have just observed involving identity, inverse and composition can be used to reveal properties of the congruence relation. Recall that two subsets S and T of a plane Π are *congruent,* denoted $S \cong T$, if there is a rigid motion of Π that moves S to T. Thus, $S \cong T$ if and only if there is a rigid motion M of Π such that $M(S) = T$. We now state three fundamental properties of the congruence relation and prove them using the properties of rigid motions observed previously.

Properties of Congruence. Let R, S and T be subsets of a plane Π. Then:

a) Congruence is *reflexive:* $S \cong S$.

b) Congruence is *symmetric:* If $S \cong T$, then $T \cong S$.

c) Congruence is *transitive:* If $R \cong S$ and $S \cong T$, then $R \cong T$.

Proof of a). The identity motion I_Π moves S to itself. In other words, $I_\Pi(S) = S$. Therefore, $S \cong S$. This proves congruence is reflexive.

Proof of b). Assume $S \cong T$. Then there is a rigid motion M of Π that moves S to T. Thus, $M(S) = T$. In this situation, the inverse M^{-1} is a rigid motion of Π that moves T back to S. Indeed, $M^{-1}(T) = M^{-1}(M(S)) = M^{-1} \circ M(S) = I_\Pi(S) = S$. Therefore, $T \cong S$. This proves congruence is symmetric.

Proof of c). Assume $R \cong S$ and $S \cong T$. Then there are rigid motions M_1 and M_2 of Π such that M_1 moves R to S and M_2 moves S to T. Thus, $M_1(R) = S$ and $M_2(S) = T$. In this situation, the composition $M_2 \circ M_1$ is a rigid motion of Π that moves R to T. Indeed, $M_2 \circ M_1(R) = M_2(M_1(R)) = M_2(S) = T$. Therefore, $R \cong T$. This proves congruence is transitive.

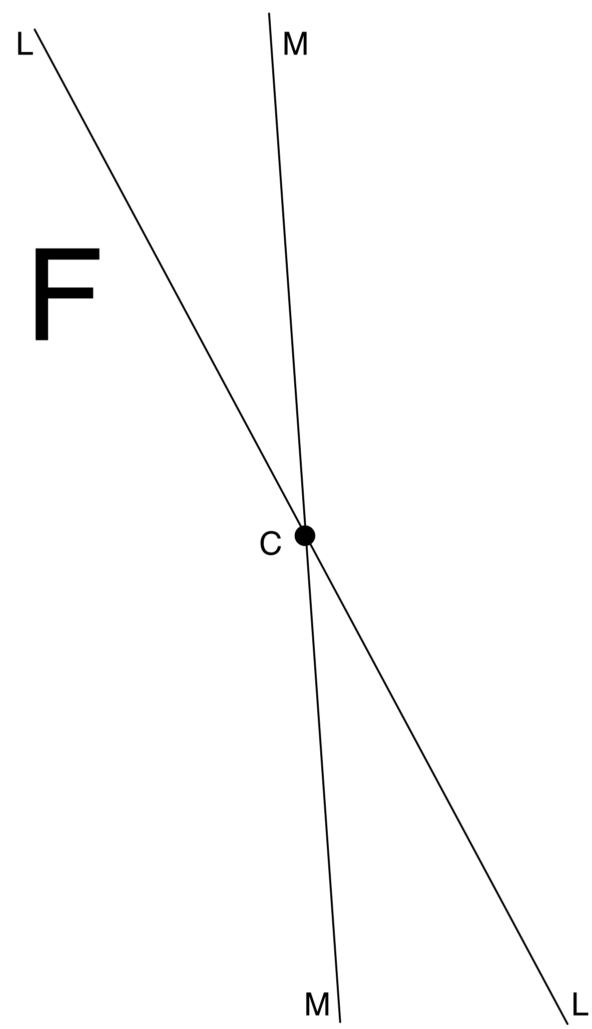
Next we begin the exploration of the types of rigid motions that result from the composition of two rigid motions. We begin with a very important case: the composition of two reflections. We also study the composition of two translations and the composition of three reflections.

Activity 2. Groups should carry out the following four activities **a)**, **b)**, **c)** and **d)** on the next four pages and report their results to the class. Use patty paper.

a) In the following picture, L and M are lines that meet at the point C. Let **F** represent the large figure F in this picture. Use patty paper to construct $Z_M \circ Z_L$(**F**). Study the relationship between **F** and $Z_M \circ Z_L$(**F**) to determine which of the four types of rigid motions $Z_M \circ Z_L$ is. Using our notation for rigid motions, fill in the blank in the equation

$$Z_M \circ Z_L \ = \ \underline{\hspace{4cm}}.$$

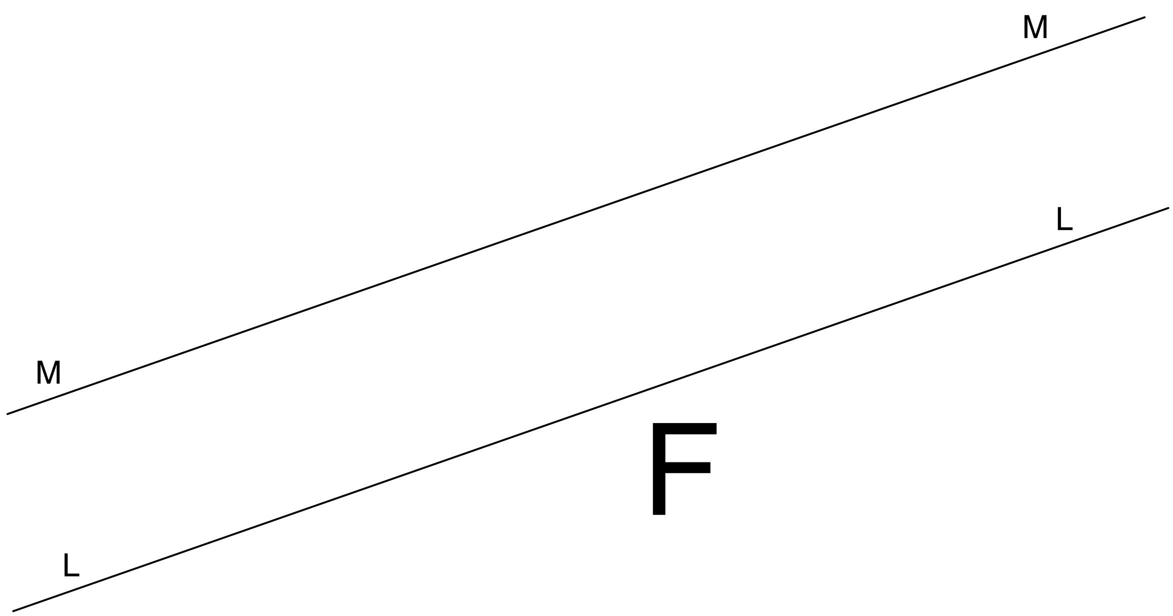
(If necessary, draw and label new points, lines or angles in the picture and use the names of these objects in your expression in the blank.)



b) In the following picture, L and M are parallel lines. Let **F** represent the large figure F in this picture. Use patty paper to construct $Z_M \circ Z_L$(**F**). Study the relationship between **F** and $Z_M \circ Z_L$(**F**) to determine which of the four types of rigid motions $Z_M \circ Z_L$ is. Using our notation for rigid motions, fill in the blank in the equation

$$Z_M \circ Z_L \; = \; \_\_\_\_\_\_\_\_\_\_\_\_\_\_ .$$

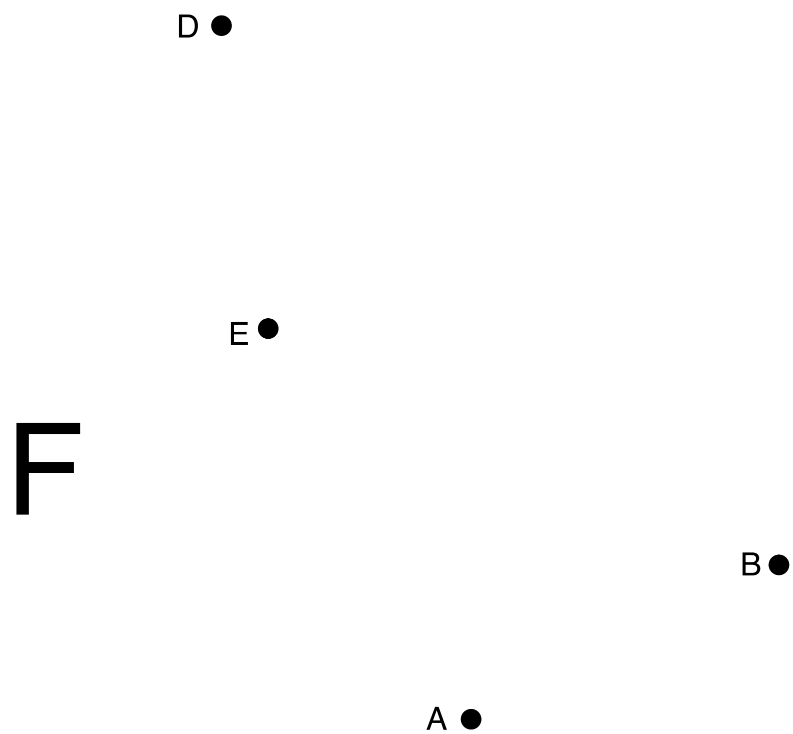
(If necessary, draw and label new points, lines or angles in the picture and use the names of these objects in your expression in the blank.)



c) In the following picture, A, B, D and E are four points. Let **F** represent the large figure F in this picture. Use patty paper to construct $T_{D,E} \circ T_{A,B}$(**F**). Study the relationship between **F** and $T_{D,E} \circ T_{A,B}$(**F**) to determine which of the four types of rigid motions $T_{D,E} \circ T_{A,B}$ is. Using our notation for rigid motions, fill in the blank in the equation

$$T_{D,E} \circ T_{A,B} = \underline{\hspace{4cm}}.$$

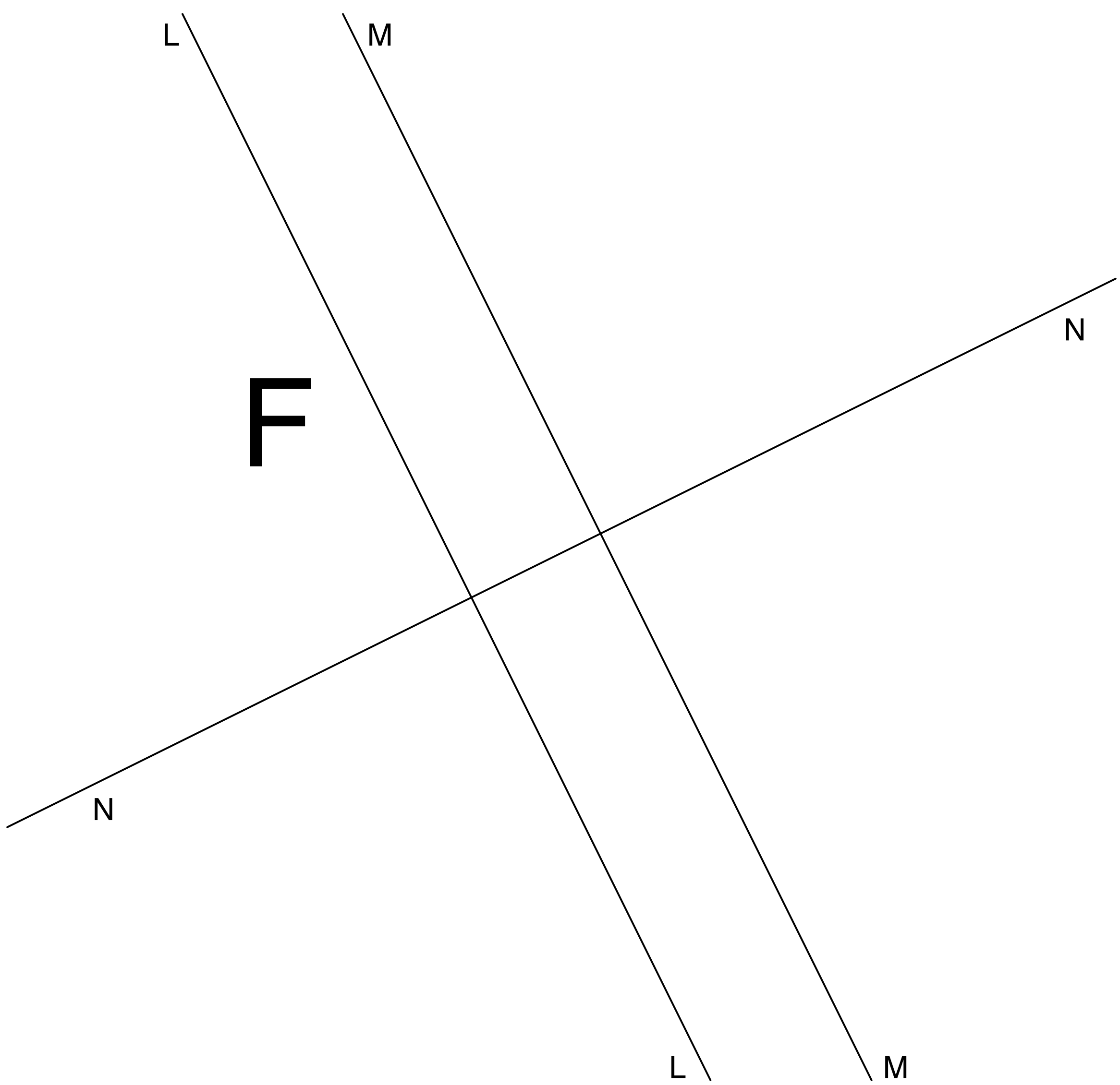
(If necessary, draw and label new points, lines or angles in the picture and use the names of these objects in your expression in the blank.)



d) In the following picture, L and M are parallel lines and N is a line that is perpendicular to L and M. Let **F** represent the large figure F in this picture. Use patty paper to construct $Z_N \circ Z_M \circ Z_L(\mathbf{F})$. Study the relationship between **F** and $Z_N \circ Z_M \circ Z_L(\mathbf{F})$ to determine which of the four types of rigid motions $Z_N \circ Z_M \circ Z_L(\mathbf{F})$ is. Using our notation for rigid motions, fill in the blank in the equation

$$Z_N \circ Z_M \circ Z_L = \underline{\hspace{4cm}}.$$

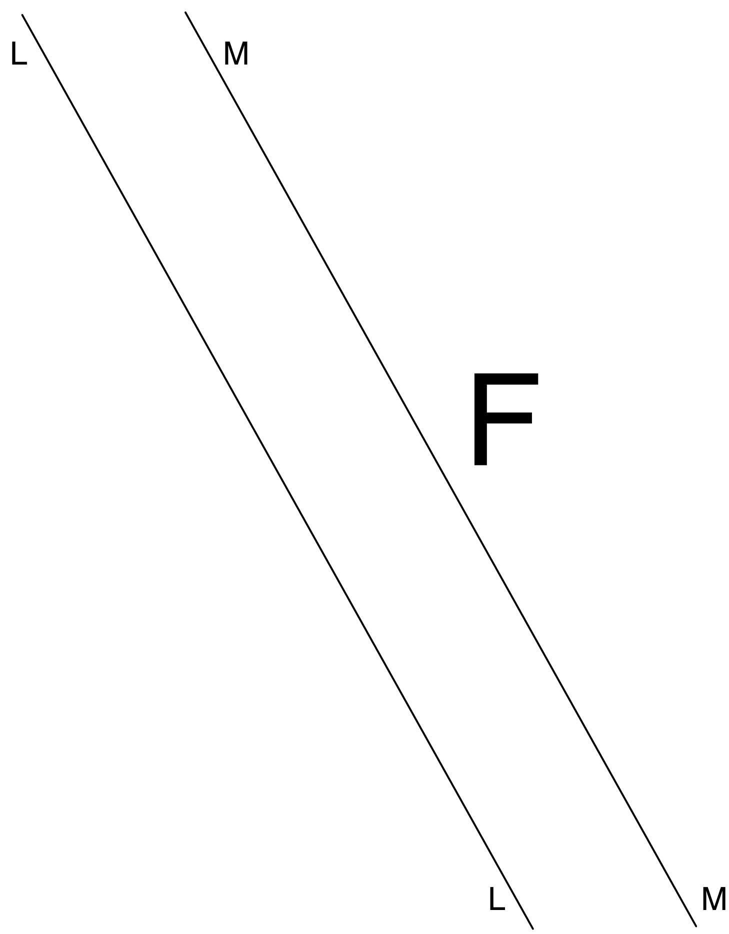
(If necessary, draw and label new points, lines or angles in the picture and use the names of these objects in your expression in the blank.)



Homework Problem 1. In the following picture, L and M are parallel lines. Let **F** represent the large figure F in this picture. Use patty paper to construct $Z_M \circ Z_L$(**F**). Study the relationship between **F** and $Z_M \circ Z_L$(**F**) to determine which of the four types of rigid motions $Z_M \circ Z_L$ is. Using our notation for rigid motions, fill in the blank in the equation

$$Z_M \circ Z_L \;=\; \_\_\_\_\_\_\_\_\_\_\_\_ .$$

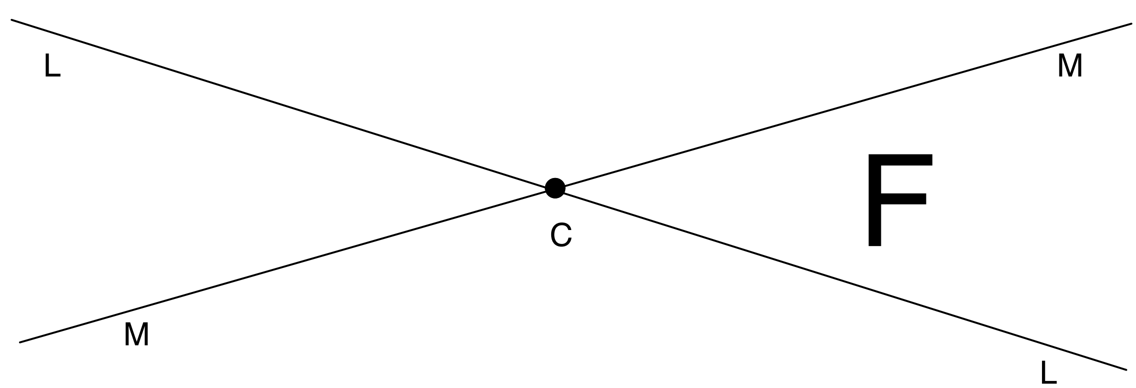
(If necessary, draw and label new points, lines or angles in the picture and use the names of these objects in your expression in the blank.)



Homework Problem 2. In the following picture, L and M are lines that meet at the point C. Let **F** represent the large figure F in this picture. Use patty paper to construct $Z_M \circ Z_L$(**F**). Study the relationship between **F** and $Z_M \circ Z_L$(**F**) to determine which of the four types of rigid motions $Z_M \circ Z_L$ is. Using our notation for rigid motions, fill in the blank in the equation

$$Z_M \circ Z_L = \underline{\hspace{3cm}}.$$

(If necessary, draw and label new points, lines or angles in the picture and use the names of these objects in your expression in the blank.)



Homework Problem 3. Activity 2 and Homework Problems 1 and 2 revealed the type of rigid motions that result from the composition of certain types of rigid motions in specific situations. This problem asks you to generalize the implications of these activities and express these generalizations in words by filling in the blanks in the following four statements.

a) If L and M are lines in a plane Π that intersect at a point C, then the composition $Z_M \circ Z_L$ is a ____.

b) If L and M are parallel lines in a plane Π, then the composition $Z_M \circ Z_L$ is a

____.

c) If L and M are parallel lines in a plane Π and N is a line in the plane Π that is perpendicular to L and M, then the composition $Z_N \circ Z_M \circ Z_L$ is a

____.

d) If A, B, C and D are four points in a plane Π, then the composition $T_{C,D} \circ T_{A,B}$ is a

____.

Homework Problem 4. In parts a) and b) of Activity 2 and in Homework Problems 1 and 2, you are asked to apply the composition $Z_M \circ Z_L$ of two reflections to a figure **F**. Return to these exercises and apply the composition $Z_L \circ Z_M$ of the same two reflections in the opposite order to the letter **F**. In these situations, observe whether it is ever the case that $Z_M \circ Z_L(\mathbf{F}) = Z_L \circ Z_M(\mathbf{F})$? From your observations, answer the following questions.

a) If L and M are lines in a plane Π that intersect at a point C, is it ever the case that $Z_M \circ Z_L = Z_L \circ Z_M$?

b) If L and M are parallel lines in a plane Π, is it ever the case that $Z_M \circ Z_L = Z_L \circ Z_M$?

c) Fill in the blank in the statement below to formulate a *theorem* that gives the exact conditions under which two reflections commute.

If L and M are lines in a plane Π, then $Z_M \circ Z_L = Z_L \circ Z_M$ if and only if:

____

Homework Problem 5. The figure below contains the solid lines L, M and N together with several unnamed dotted lines, and points labeled A, B, C, D, E, F, G and H. Let M denote the rigid motion which is the composition of the three reflections Z_L, Z_M and Z_N:

$$M = Z_N \circ Z_M \circ Z_L.$$

Construct and label the points M(A), M(B), M(C), M(D), M(E), M(F), M(G) and M(H) in this figure. Study the relationship between the points A, B, C, D, E, F, G and H and their *images* M(A), M(B), M(C), M(D), M(E), M(F), M(G) and M(H) to determine which of the four types of rigid motions M is. Using our notation for rigid motions, fill in the blank in the equation

$$Z_N \circ Z_M \circ Z_L = \underline{\qquad\qquad\qquad}.$$

(If necessary, draw and label new points, lines or angles in the picture and use the names of these objects in your expression in the blank.)