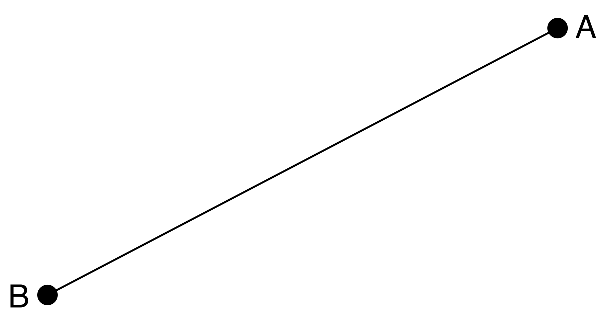
Lesson 7: Patty Paper Constructions and Rigid Motions

This lesson begins an exploration of rigid motions of planes and the symmetry of plane figures. Many of the concepts that we will study can be generalized to rigid motions of 3-dimensional space and the symmetry of solid figures, but time won't permit us to explore this.

One of our principal tools for studying rigid motions and symmetry is *patty paper.*[1] We begin by developing techniques for using patty paper to carry out geometric constructions.

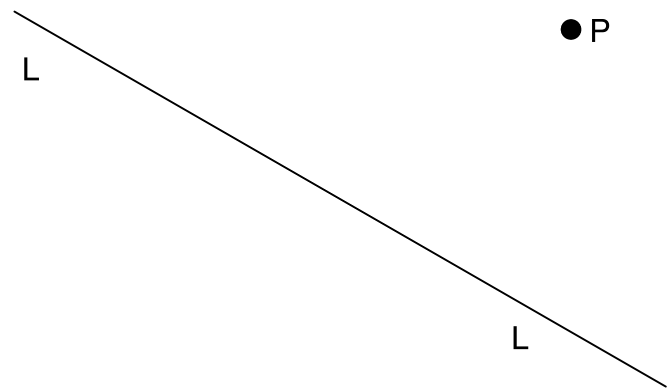
Activity 1. Groups should carry out this activity and report their results to the class. In parts a) through d) of this activity on this and the next two pages, points, lines, line segments and angles are given. You must devise techniques that use a pencil, a ruler and a piece of patty paper to carry out the indicated geometric constructions.

a) Construct the perpendicular bisector of the given line segment $\overline{AB}$. (The *perpendicular bisector* of $\overline{AB}$ is the line that is perpendicular to $\overline{AB}$ and passes through the midpoint of $\overline{AB}$.)

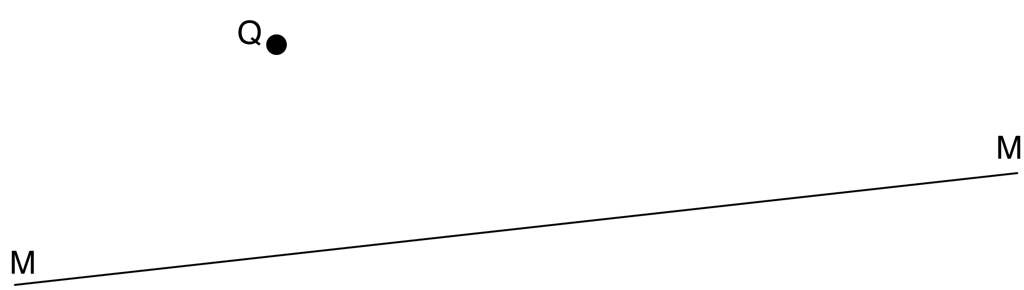


[1] The program *Cabri Jr.* on a hand-held calculator performs similar functions.

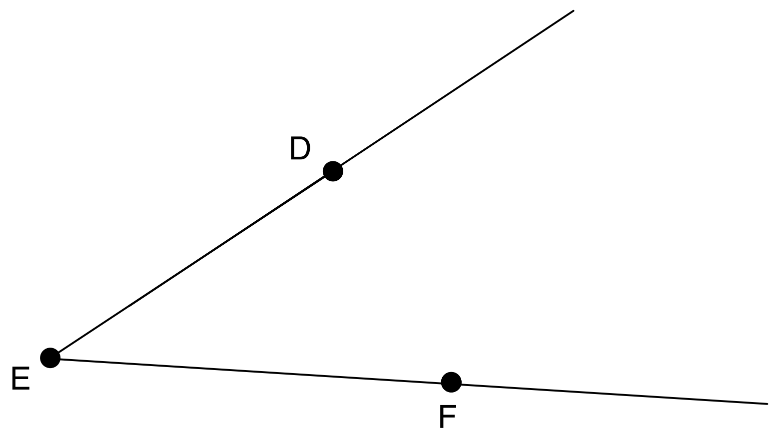
b) Drop the perpendicular from the given point P to the given line L.



c) Construct the line through the given point Q that is parallel to the given line M.



d) Construct the bisector of the given angle $\angle DEF$.



Remark. Although patty paper construction is “low tech”, it is none the less a serious geometric technique. As a construction technique, it is at least as powerful as the technique of ruler and compass construction.

We now begin our exploration of the subject of rigid motions. Our first step is to introduce notation for some fundamental geometric concepts that will allow us to define and discuss rigid motions.

Notation. Let A and B be points in space.

- If $A \neq B$, let $\overleftrightarrow{AB}$ denote the unique line determined by A and B.
- Let $\overline{AB}$ denote the *line segment* with endpoints A and B. Thus, $\overline{AB}$ consists of all the points of the line $\overleftrightarrow{AB}$ that lie between A and B, together with the points A and B themselves.
- If $A \neq B$, let $\overrightarrow{AB}$ denote the *ray* that emanates from A and passes through B. Thus, $\overrightarrow{AB}$ consists of all the points of the line $\overleftrightarrow{AB}$ that lie either between A and B or on the opposite side of B from A, together with the points A and B themselves.
- Let AB denote the distance between A and B.

We make several observations about this notation. First note that if A and B are points in space, then AB is a *number* which is ≥ 0, while $\overleftrightarrow{AB}$, $\overline{AB}$ and $\overrightarrow{AB}$ are *sets* in space. Indeed, AB is the *length* of $\overline{AB}$. **Warning:** Don't write one of these expressions when you mean the other. Also, we recall that distance has the following three properties:

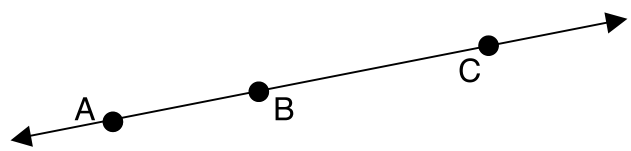
$$AB \geq 0, \quad AB = 0 \text{ if and only if } A = B, \quad \text{and} \quad AB = BA.$$

Definition. Let A, B and C be three points in a plane or in space. If A, B and C all lie on the same line, we say they are *collinear.* If A, B and C do not lie on a single line, then we say they are *noncollinear.*

Here is a basic fact concerning the concepts just introduced.

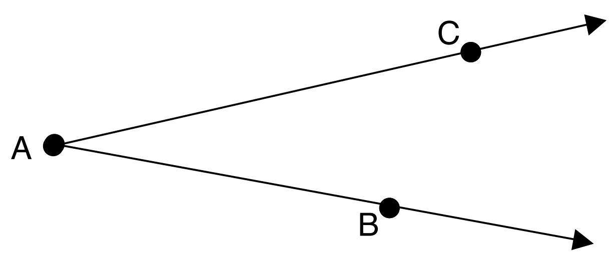
The Length Addition Axiom. If A, B and C are collinear points and B lies between A and C, then

$$AC = AB + BC.$$



Next we define the concept of *angle.*

Definition. Let A, B and C be three points in space such that A ≠ B and A ≠ C. The union of the two rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ is called the *angle* with *vertex* A and *sides* $\overrightarrow{AB}$ and $\overrightarrow{AC}$. This angle can be denoted either ∠BAC or ∠CAB.

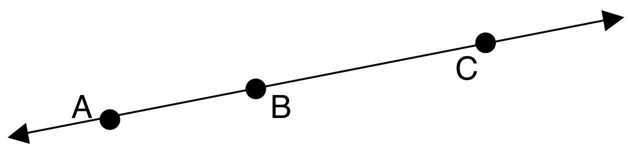


∠BAC

We assume that you understand that every angle ∠BAC has a *measure* that is a number between 0 and 180. The measure of the angle ∠BAC is denoted m(∠BAC). To indicate that angles are being measured in *degrees,* we attach the symbol "°" to the numerical measure of each angle. Thus, $0° \leq m(\angle BAC) \leq 180°$.

We observe that if ∠BAC is an angle in space, then ∠BAC is a *set* in space, while m(∠BAC) is a *number* between 0º and 180º. **Warning:** Don't write one of these when you mean the other.

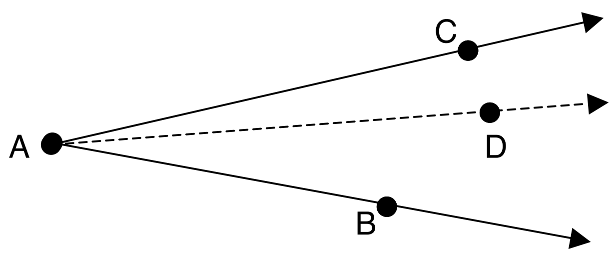
Recall that if A, B and C are collinear points, and B lies between A and C, then the angle ∠ABC is often called a *straight angle.*



Here are two basic facts about angles.

The Angle Addition Axiom. If D is a point that lies in the interior of the angle ∠BAC, then

$$m(\angle BAC) = m(\angle BAD) + m(\angle DAC).$$



The Straight Angle Axiom. If A, B and C collinear points and B is between A and C, then

$$m(\angle ABC) = 180^\circ.$$

Definition. A *rigid motion* of a plane is a motion of points of the plane to other points of the plane that preserves the distance between any two points. Thus, M is a *rigid motion* of a plane ∏ if:

- M moves each point A of the plane ∏ to another point M(A) of the plane ∏, and
- if A and B are any two points of the plane ∏, then the distance from A to B equals the distance from M(A) to M(B). (In terms of our notation: AB = M(A)M(B).)

(∏ is the upper case Greek alphabet letter whose name is "pi".)

We remark that a rigid motion of space can be defined similarly. It is a distance preserving motion of the points of space.

Note that we are using *function notation* to write about rigid motions. In function notation, if a rigid motion of a plane is named M and a point of the plane is named A, then the point to which the rigid motion M moves A is denoted M(A). Similarly, a rigid motion named T would move the point named B to the point denoted T(B). We also use function notation to denote the result of applying a rigid motion to a set: if M is a rigid motion of a plane and S is a set in the plane, then the set to which the rigid motion M moves S is denoted M(S).

Now that we have introduced the concept of a *rigid motion*, we are in a position to define the most important idea in geometry: *congruence.*

Definition. Two geometric figures in a plane are *congruent* if and only if there is a rigid motion of the plane that moves one figure to the other.

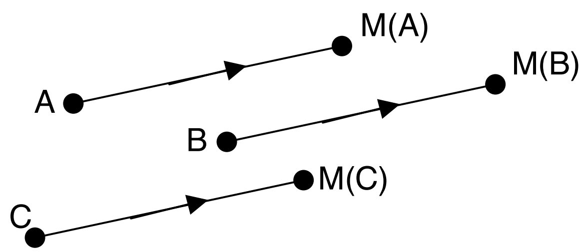
We remark that we can similarly define the relation of congruence for two geometric figures in space. Two geometric figures in space are *congruent* if there is a rigid motion of space that moves one figure to the other.

Notation. If S and T are geometric figures in a plane Π, then we will write $S \cong T$ to indicate that S is congruent to T. Thus, $S \cong T$ means there is a rigid motion of Π that moves S to T. In other words, S is congruent to T if and only if there is a rigid motion M of Π such that M(S) = T.

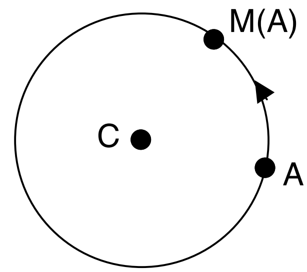
A rigid motion of a plane moves the plane onto itself *rigidly*. It doesn't bend, fold, stretch, shrink or tear the plane. It acts as if the plane where a stiff piece of wood or metal. The rigidity of the motion is expressed mathematically by saying that the motion preserves distances between points. Another name for a rigid motion is an *isometry.* The word *isometry* is Greek for "same distance".

To enrich your understanding of rigid motions, we describe four different types of rigid motions of a plane.

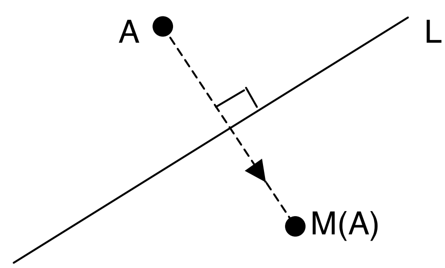
Type 1: Translation. A *translation* moves all the points of the plane in the same direction through the same distance.



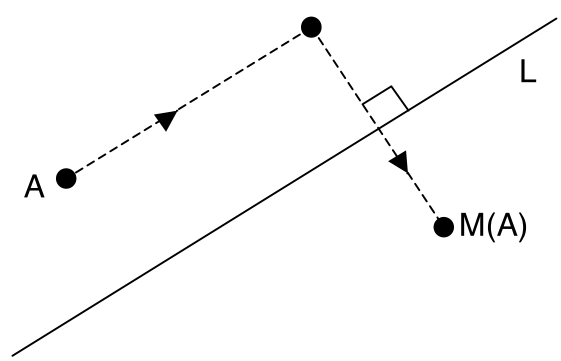
Type 2: Rotation. A *rotation* fixes one point C and rotates every other point of the plane about C through the same angle.



Type 3: Reflection. A *reflection* fixes all the points of a line L, and it moves each point A that lies on one side of L to a point M(A) lying on the other side of L so that A and M(A) determine a line that is perpendicular to L and A and M(A) are equidistant from L.



Type 4: Glide Reflection. A *glide reflection* is a rigid motion that results from first performing a translation in a direction that is parallel to a line L and then performing a reflection across the line L.



Remarkably, every rigid motion of a plane is of one of these four types. We state this important fact in the form of a theorem.

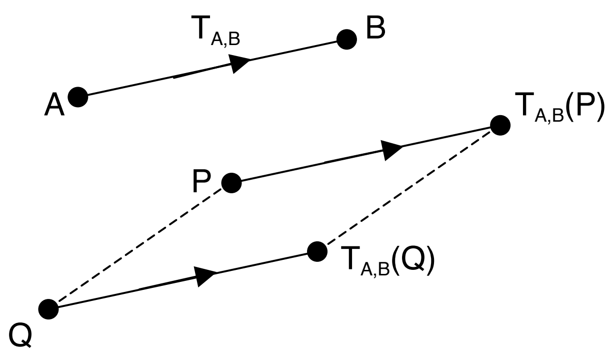
The Classification Theorem for Rigid Motions of a Plane. Every rigid motion of a plane is either a translation, a rotation, a reflection or a glide reflection.

This is a powerful and useful theorem which we will unfortunately not have time to prove. However, we will use it to draw various conclusions in subsequent lessons.

We now introduce some useful notation for these four types of rigid motions, and observe some important properties of each type of rigid motion.

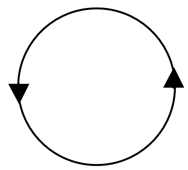
Notation for and Properties of the Four Types of Rigid Motions

Translations. If A and B are points in a plane ∏, then we let $T_{A,B}$ denote the unique translation of ∏ that moves the point A to the point B. Thus, if P is any point of the plane ∏, then $T_{A,B}$ moves P to the point $T_{A,B}(P)$. In particular, $T_{A,B}(A) = B$.

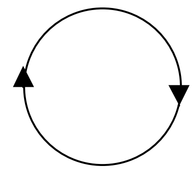


Observation. If P and Q are distinct points of the plane ∏, notice that the quadrilateral with vertices P, $T_{A,B}(P)$, $T_{A,B}(Q)$ and Q is a parallelogram. (Recall that a quadrilateral is a *parallelogram* if its opposite sides are parallel.

Rotations. Let C be a point in a plane ∏. Every angle in the plane ∏ with vertex C has two possible *orientations: counterclockwise* and *clockwise*. An angle to which an

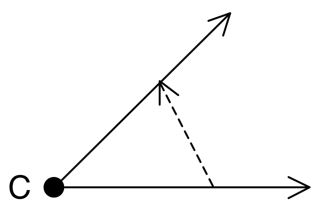


Counterclockwise

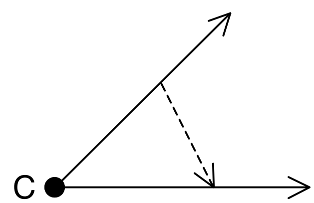


Clockwise

orientation has been assigned is called an *oriented angle.* We indicate the orientation of an angle in a figure by drawing a dotted arrow.

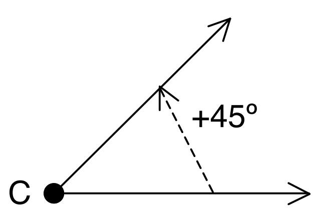


Counterclockwise
oriented angle

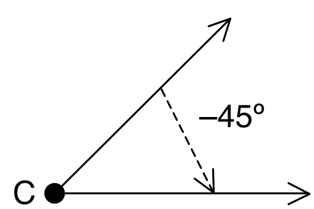


Clockwise
oriented angle

Every oriented angle has an *oriented measure.* The measure of an oriented angle has the same absolute value as the measure of the corresponding unoriented angle. But the measure of the oriented angle is positive if the angle is oriented counterclockwise and negative if the angle is oriented clockwise.

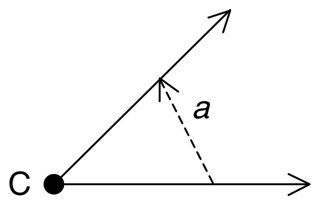


Counterclockwise
oriented angle with
oriented measure +45°

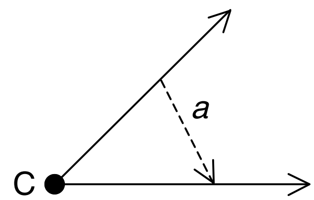


Clockwise
oriented angle with
oriented measure –45°

We will illustrate an oriented angle with oriented measure a by drawing an angle with dotted arrow to indicate orientation and the letter a written next to the dotted arrow to indicate the measure of the oriented angle. If the dotted arrow shows a counterclockwise orientation, then a represents a positive number; and if the dotted arrow shows a clockwise orientation, then a represents a negative number.

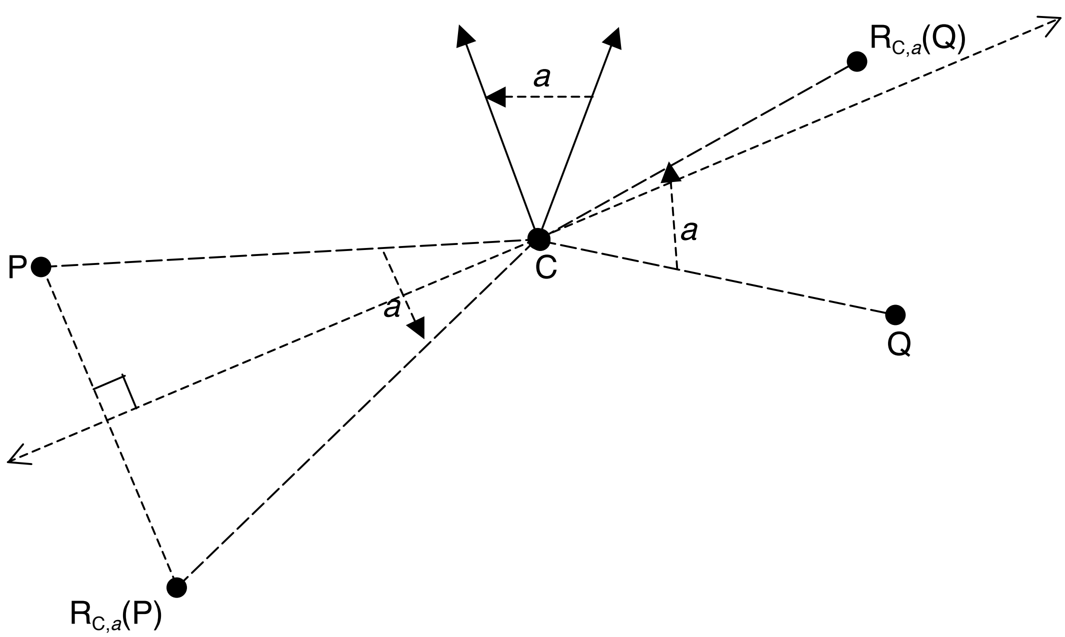


Here the orientation is counterclockwise and $a > 0$.



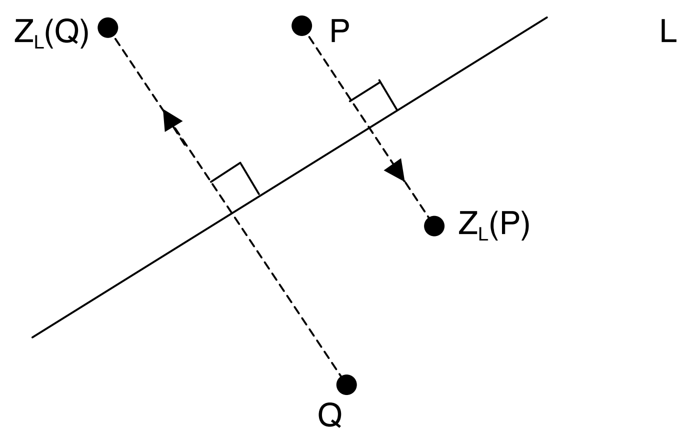
Here the orientation is clockwise and $a < 0$.

Now let C be a point in a plane Π, and let a be the oriented measure of an oriented angle in the plane Π with vertex C. ($a > 0$ if the angle is oriented counterclockwise, and $a < 0$ if the angle is oriented clockwise.) Then we let $R_{C,a}$ denote the unique rotation that fixes the point C and rotates every other point of the plane Π around C through an oriented angle of measure a. We call the point C the *center* of the rotation $R_{C,a}$. Thus, if P is any point of the plane Π, then $R_{C,a}$ moves P to the point $R_{C,a}(P)$. In particular, $R_{C,a}(C) = C$.



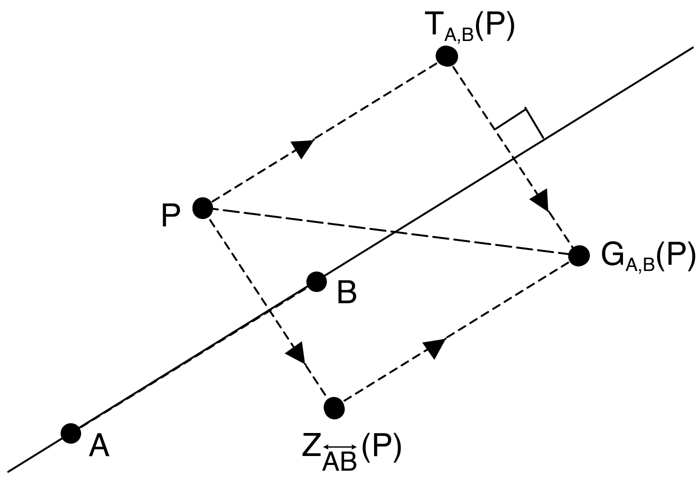
Observation. Let P be any point of the plane ∏ other than C. In the previous figure, find the oriented angle $\angle PC(R_{C,a}(P))$. Notice that this angle has oriented measure *a*. Also notice that the perpendicular bisector of the line segment $\overline{P(R_{C,a}(P))}$ passes through the point C.

Reflections. If L is a line in a plane ∏, then we let Z_L denote the unique reflection across the line L. We call the line L the *line of reflection* of Z_L. Thus, if P is any point of the plane ∏, then Z_L moves P to the point $Z_L(P)$. In particular, if the point P lies on the line L, then $Z_L(P) = P$.



Observation. If P is any point of the plane ∏ that does not lie on the line L, then notice that the line L is the perpendicular bisector of the line segment $\overline{P(Z_L(P))}$.

Glide Reflections. If A and B are points in a plane Π, we let $G_{A,B}$ denote the unique glide reflection of Π that translates the point A to the point B and then reflects every point of Π across the line $\overleftrightarrow{AB}$. We call the line $\overleftrightarrow{AB}$ the *line of reflection* of $G_{A,B}$. Thus, if P is any point of the plane Π, then $G_{A,B}$ moves P to the point $G_{A,B}(P)$. In particular, $G_{A,B}(A) = B$.



Observation. Notice that performing the glide reflection $G_{A,B}$ has the same effect as first performing the translation $T_{A,B}$ and then performing the reflection $Z_{\overleftrightarrow{AB}}$ across the line $\overleftrightarrow{AB}$. (Also, $G_{A,B}$ has the same effect as first performing the reflection $Z_{\overleftrightarrow{AB}}$ translation $T_{A,B}$ and then performing the translation $T_{A,B}$. In other words, the glide reflections $G_{A,B}$ results from performing the translation $T_{A,B}$ and the reflection $Z_{\overleftrightarrow{AB}}$ *in either order.*) Also notice that if P is point in the plane Π that does not lie on the line $\overleftrightarrow{AB}$, then the quadrilateral with vertices P, $T_{A,B}(P)$, $G_{A,B}(P)$ and $Z_{\overleftrightarrow{AB}}(P)$ is a rectangle. Furthermore, the line $\overleftrightarrow{AB}$ bisects two sides of this rectangle and is parallel to the other two sides. Also the line $\overleftrightarrow{AB}$ bisects the diagonal $\overline{P(G_{A,B}(P))}$ of this rectangle.