Lesson 5: The Height and Distance Lemmas

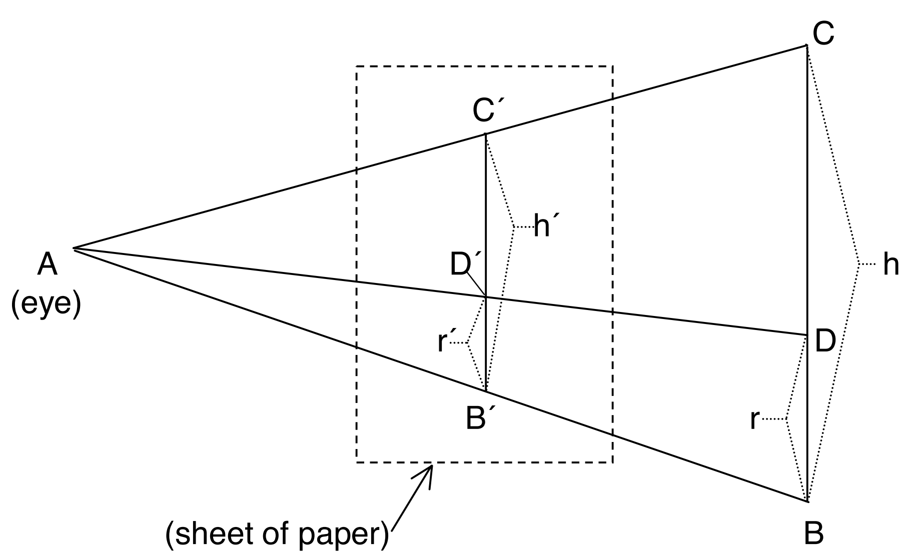
The activities and homework problems in this lesson ask us to compute large scale measurements (such as the height of a building) from pictures sketched by an observer. The pictures contain lines drawn from the observer's eye toward prominent features of the object being measured (such as the top of the building). To make these computations easy, we will state two theorems that apply specifically to the types of computations we will be making. The theorems are called the *Height Lemma* and the *Distance Lemma.* We will state these lemmas, prove the Height Lemma, and ask you to prove the Distance Lemma in Homework Problem 3. These proofs depend on the three theorems about similar triangles stated in Lesson 4.

The Height Lemma. Suppose that A, B, B´, C, C´, D, and D´ are points such that

- A, B, and B´ are collinear with B´ between A and B,
- A, C, and C´ are collinear with C´ between A and C,
- A, D, and D´ are collinear with D´ between A and D,
- D lies on $\overline{BC}$,
- D´ lies on $\overline{B'C'}$, and
- $\Delta ABC \sim \Delta AB'C'$.

Let $h = BC$, $r = BD$, $h' = B'C'$, $r' = B'D'$.

Then $h = \left(\dfrac{h'}{r'}\right) r$.



When we apply the Height Lemma, we will be in a situation where h is a height we want to measure and r is a *reference height* that we already know. The observer's eye is at the point A, and the lengths labeled h´ and r´ are drawn on a sheet of paper and are easy to measure with a ruler.

The Distance Lemma. Suppose that A, B, B´, C, and C´ are points such that

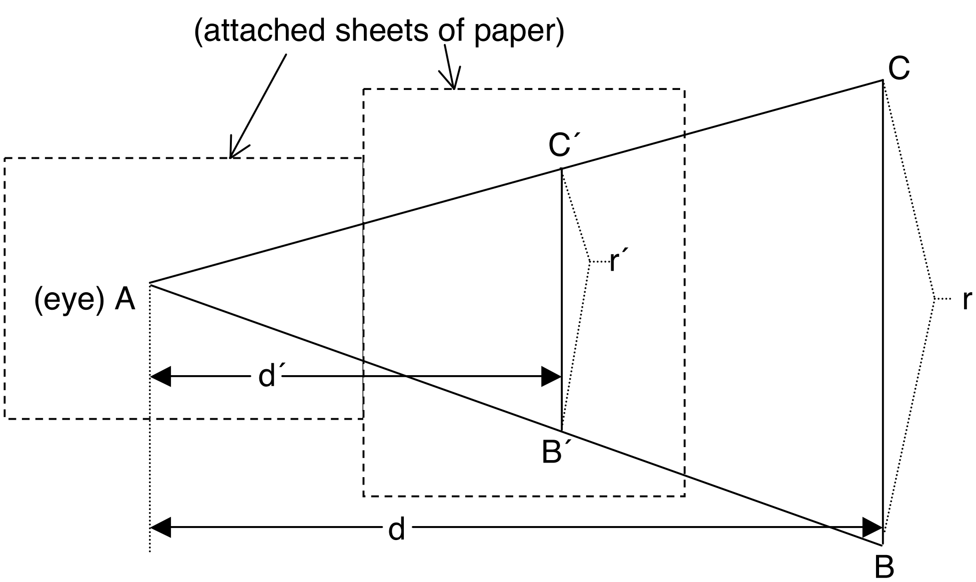
- A, B, and B´ are collinear with B´ between A and B,
- A, C, and C´ are collinear with C´ between A and C, and
- $\Delta ABC \sim \Delta\, AB'C'$.

Let

d = the distance from A to $\overline{BC}$, r = BC

d' = the distance from A to $\overline{B'C'}$, r' = B'C'.

Then $d = \left(\dfrac{d'}{r'}\right) r$.



When we apply the Distance Lemma, we will be in a situation where d is a distance we want to measure and r is a *reference height* that we already know. The observer's eye is at the point A, and the lengths labeled d´ and r´ are drawn on attached sheets of paper and are easy to measure with a ruler.

We will now present a convincing explanation (in other words, a *proof*) of the Height Lemma. The proof relies on the three theorems about similar triangles stated in Lesson 4. Since the proof of the Height Lemma is a little indirect, we first present a *motivation* for the proof.

Motivation for the proof of the Height Lemma. Look at the figure accompanying the statement of the Height Lemma. It suffices to prove that the two ratios $\frac{h}{h'}$ and $\frac{r}{r'}$ are equal. (**Why?**) $\frac{h}{h'}$ is the ratio between the lengths of corresponding sides in the two big triangles ΔABC and ΔAB´C´ (because h = BC and h´ = B´C´). $\frac{r}{r'}$ is the ratio between the lengths of corresponding sides in the two smaller triangles ΔABD and ΔAB´D´ (because r = BD and r´ = B´D´). The trick to establishing an equality between $\frac{h}{h'}$ and $\frac{r}{r'}$ is to show that both of these ratios are equal to a *third ratio* that appears as a ratio between corresponding side lengths in *both* pairs of triangles. If we examine the two pairs of triangles – the pair ΔABC and ΔAB´C´, and the pair ΔABD and ΔAB´D´ – we discover that there is a ratio between lengths of corresponding sides that is common to both these pairs of triangles, namely $\frac{AB}{AB'}$. So our goal is to prove that the two ratios $\frac{h}{h'}$ and $\frac{r}{r'}$ are both equal to $\frac{AB}{AB'}$. One of the *hypotheses* (given conditions) of the Height Lemma is the similarity relation ΔABC ~ ΔAB´C´. The equation $\frac{h}{h'} = \frac{AB}{AB'}$ follows from this similarity relation. If we also knew the similarity relation ΔABD ~ Δ AB´D´, then we would be able to conclude from that relation that the equation $\frac{r}{r'} = \frac{AB}{AB'}$ holds. This would complete the proof. Unfortunately, the similarity relation ΔABD ~ ΔAB´D´ is *not* a given hypothesis of the Height Lemma. In order to make our approach to this proof work, we must establish the similarity relation ΔABD ~ ΔAB´D´ as part of the proof. Fortunately, this is possible by using the Angle Theorem for Similar Triangles. Specifically, we can demonstrate that the measures of the angles at A and B in ΔABD are equal to the measures of the angles at A and B´ in ΔAB´D´. This ends the motivation. We now turn to the actual proof of the Height Lemma.

Proof of the Height Lemma. The similarity relation ΔABC ~ ΔAB´C´ is given. Therefore the Side Theorem for Similar Triangles implies $\frac{h}{h'} = \frac{AB}{AB'}$.

Next we prove ΔABD ~ ΔAB´D´. Note that the angle at A in ΔABD is identical to the angle at A in ΔAB´D´; therefore, these two angles have equal measures. Since ΔABC ~ ΔAB´C´, then the Angle Theorem for Similar Triangles implies that the angle at B in ΔABC and the angle at B´ in ΔAB´C´ have equal measures. Note that the angle at B in ΔABC and the angle at B in ΔABD are identical; therefore, these two angles have equal measures. Similarly, since that the angle at B´ in ΔAB´C´ and the angle at B´ in ΔAB´D´ are identical, then these two angles have equal measures. Combining the results of the previous three sentences, we conclude that the angle at B in ΔABD and the angle at B´ in ΔAB´D´ have equal measures. We summarize what we have just discovered:

- The angle at A in ΔABD and the angle at A in ΔAB´D´ have equal measures.
- The angle at B in ΔABD and the angle at B´ in ΔAB´D´ have equal measures.

These two facts together with the Angle Theorem for Similar Triangles imply the similarity relation ΔABD ~ ΔAB´D´.

The similarity relation ΔABD ~ ΔAB´D´ together with the Side Theorem for Similar Triangles imply $\frac{r}{r'} = \frac{AB}{AB'}$.

We have established the two equations $\frac{h}{h'} = \frac{AB}{AB'}$ and $\frac{r}{r'} = \frac{AB}{AB'}$. These two equations imply $\frac{h}{h'} = \frac{r}{r'}$. A little algebra converts this equation to: $h = \left(\frac{h'}{r'}\right)r$. This completes the proof of the Height Lemma.

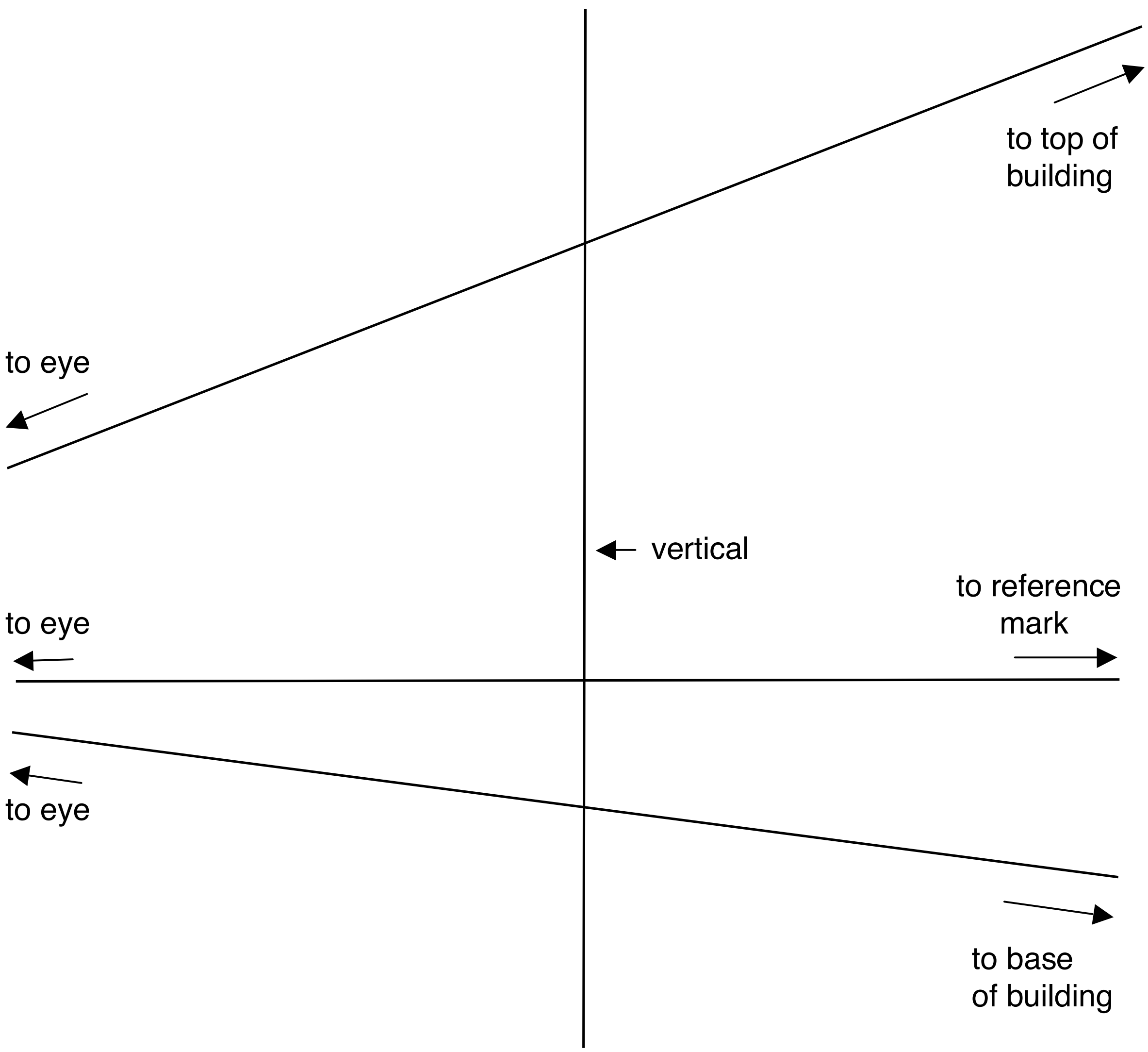
Activity 1. Groups should read and discuss Homework Problem 1 from Lesson 5 (below) and report their results to the class.

Activity 2. Groups should read and discuss Homework Problem 2 from Lesson 5 (below) and report their results to the class.

Homework Problem 1: Dan's Problem. Dan wants to measure the height of a building. The building has a reference mark on it that is 3.0 meters above the ground. So he stands facing the reference mark on the building and holds a clipboard with a piece of paper on it vertically so that he can draw lines on the paper that run from his eye to various points on the building. He draws four lines on the paper:

- one line from his eye to the base of the building,
- one line from his eye to the reference mark on the building,
- one line from his eye to the top of the building, and
- one vertical line.

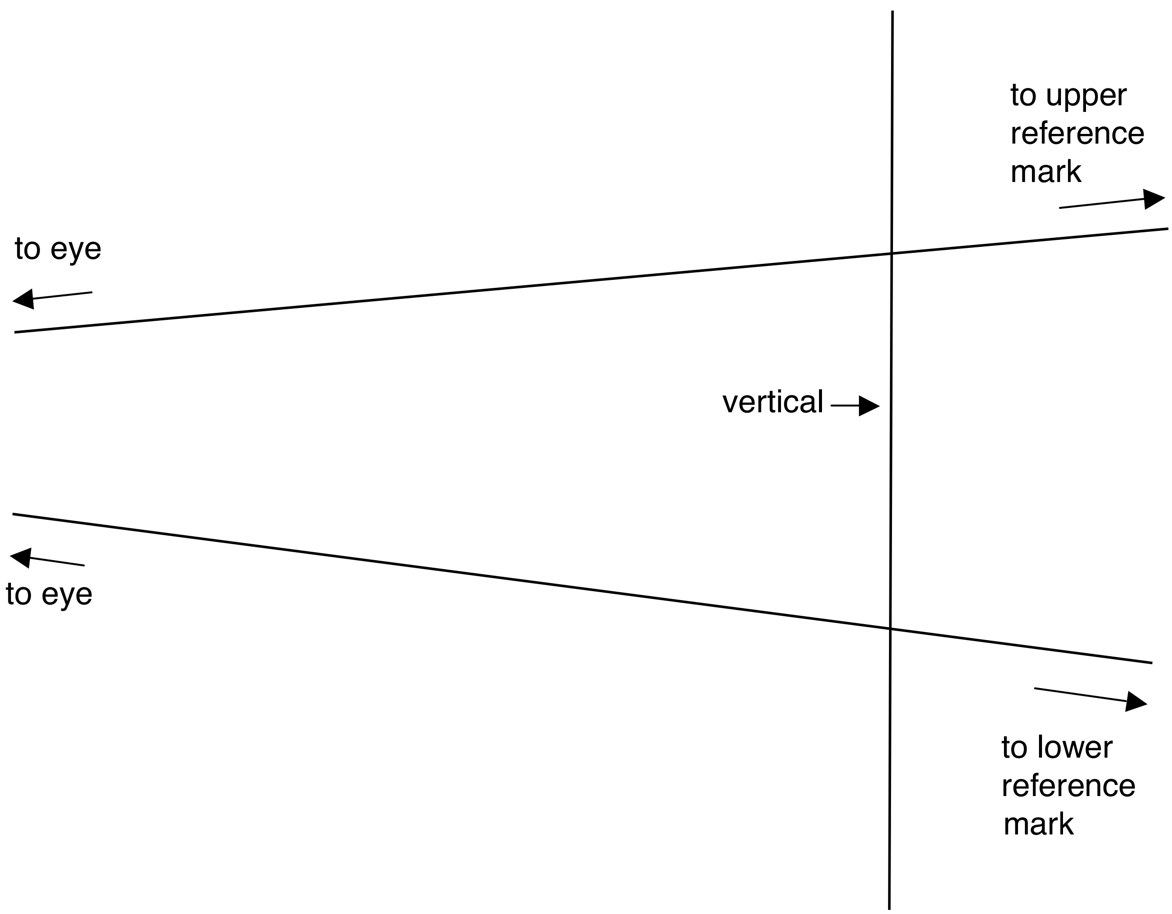
These lines are shown in the figure below. He then measures some distances on the piece of paper and uses the *Height Lemma* to calculate the height of the building. What answer do you think he should get?



Homework Problem 2: Hillary's Problem. Hillary wants to measure the distance from the point where she is standing to a flagpole that has two reference marks on it that she knows are 4.0 feet apart. So she stands facing flagpole and holds a clipboard with a piece of paper on it vertically so that she can draw lines on the paper that run from her eye various points on the flagpole. She draws three lines on the paper:

- one line from her eye to the lower reference mark on the flagpole,
- one line from her eye to the upper reference mark on the flagpole, and
- one vertical line.

These lines are shown in the figure below. She then tapes a piece of paper to the left edge of this paper so that she can complete the triangle. She then measures some distances on the taped together pieces of paper and uses the *Distance Lemma* to calculate her distance from the flagpole. What answer do you think she should get?



Homework Problem 3. Find a proof of the *Distance Lemma.* Use the three theorems about similar triangles stated in Lesson 4.

Addendum to Lesson 5. To solve Homework Problem 2 above as well as Homework Problem 7 (Sharonda's Problem) in Lesson 4, it was suggested that a piece of paper be taped to the left edge of the page so that the given drawing can be extended to a point where two lines meet. In principle, if the two lines were nearly parallel, reaching the point where the lines meet could require the attachment of many pieces of paper. However, there is a "high tech" version of the *Distance Lemma* that makes it unnecessary to engage in the potentially cumbersome physical process of attaching pieces of paper. We now state and discuss this variation of the *Distance Lemma.*

The Generalized Distance Lemma. Suppose A, B, B´, B´´, C, C´, and C´´ are points such that

- A, B, B´, and B´´ are collinear with B´ and B´´ between A and B, and B´´ closer to A,
- A, C, C´, and C´´ are collinear with C´ and C´´ between A and C, and C´´ closer to A,

and

- ΔABC ~ ΔAB´C´ ~ ΔAB´´C´´.

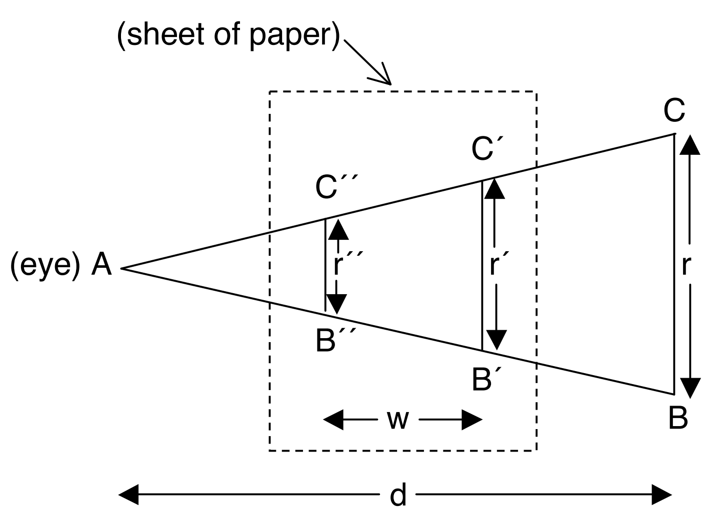
Let

d = the distance from A to $\overline{BC}$, w = the distance between $\overline{B'C'}$ and $\overline{B''C''}$,

$r = BC$, $r' = B'C'$ and $r'' = B''C''$

Then

$$d = \left(\frac{w}{r'-r''}\right)r.$$



When we apply the Generalized Distance Lemma, we will be in a situation where d is a distance we want to measure and r is a *reference height* that we already know. The observer's eye is at the point A, and the lengths labeled r´, r´´ and w are drawn on a sheet of paper and are easy to measure with a ruler.

To apply the Generalized Distance Lemma in Homework Problem 2 above, draw a second vertical line near the left edge of the page in the given figure. Then measure the distance w between the two vertical lines as well as the distances r´ and r´´ along the two vertical lines between the two lines that cross the figure from left to right. Let r be the 4 foot distance between the two reference marks on the flagpole. Then apply the formula in the *Generalized Distance Lemma.* This will allow us to compute the distance d from Hillary to the flagpole without attaching additional sheets of paper to the left edge of the page.

Homework Problem 4. Use the *Generalized Distance Lemma* to solve Homework Problem 2 without taping additional sheets of paper to the left edge of the page. Compare your result to the result you obtained by attaching a sheet of paper to the left edge of the page.

Homework Problem 5. Prove the *Generalized Distance Lemma*. In your proof you may use the fact that the *Distance Lemma* is true. You may also use the following fact.

The Subtraction Lemma. If d´, r´, d´´ and r´´ are non-zero numbers such that $d' \neq d''$ and $r' \neq r''$, and $\frac{d'}{r'} = \frac{d''}{r''}$, then

$$\frac{d'}{r'} = \frac{d' - d''}{r' - r''}.$$

Homework Problem 6. The Subtraction Lemma has a purely algebraic proof. Find it.

Homework Problem 7. Use the *Generalized Distance Lemma* to solve Homework Problem 8 (Kristin's Problem) in Lesson 4 without extending the two non-vertical lines until they meet. Compare your result to the result you obtained by extending the two non-vertical lines until they meet.