Lesson 4: Large-Scale Measurements and More About Similar Triangles

Activity 1. The class will discuss the following question. Suppose that you and a friend are standing in a large parking lot, which is featureless except for two light poles, and you wish to measure the distance between the two light poles. You have only a 3-meter measuring tape, but the distance between the two poles is much greater than 3 meters. How can you measure the distance between the poles?

Activity 2. The groups will discuss the following question and report their results to the class. Suppose that a line L is painted across a giant and otherwise featureless parking lot. You are standing at a point P in the middle of the parking lot. The point P is some distance away from the line L. You have a 3-meter measuring tape and you are accompanied by three friends who are holding the vertices of a 3-foot-by-4-foot-by-5-foot rope triangle and are free to move about the parking lot. How can you and your friends use this equipment to:

a) drop a perpendicular form P to L, and

b) measure the distance from P to L?

Activity 3. The class will view a large classroom or the lobby of the building and will discuss methods for measuring the length of the diagonal of the lobby. The class should come up with at least one method that doesn't rely on floor tiles or walls. (Hint: Recall Activity 1.) Each group should then be assigned a different method for measuring the lobby's diagonal, and it should carry out the measurement by the assigned method. (If your measurement process involves calculation, remember to use the *Calculation Rules* to avoid unnecessary inaccuracy.) The groups should then report their results to the class. The class will compare and discuss these measurements and try to account for any differences between them.

Some Geometry Notation

We establish some geometric notation that we will use for the rest of the course.

Let A and B be points in a plane or in space. Let *AB* denote the *distance* between A and B. We list three basic properties of distance.

1) $AB \geq 0$.

2) $AB = 0$ if and only if $A = B$.

3) $AB = BA$.

Let $\overline{AB}$ denote the line segment joining the points A and B. Then $\overline{AB}$ is a *set* in a plane or space, whereas AB is a non-negative real *number*. Furthermore, the number AB is the *length* of the line segment $\overline{AB}$.

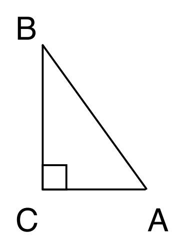
Warning: Be careful to distinguish between $\overline{AB}$ and AB. Don't write one when you mean the other. Remember: $\overline{AB}$ is a set while AB is a number.

Let A, B and C be three points in a plane or in space. If A, B and C all lie on the same line, we say they are *collinear.* If A, B and C do not lie on a single, then we say they are *noncollinear.*

If A, B and C are three noncollinear points in a plane or in space, then they are the *vertices* (or *corners*) of a *triangle.* This triangle is denoted *ΔABC.* This triangle is the *union* of the three line segments $\overline{AB}$, $\overline{AC}$ and $\overline{BC}$. The line segments $\overline{AB}$, $\overline{AC}$ and $\overline{BC}$ are called the *sides* of the triangle ΔABC. Thus, the three numbers AB, AC and BC are the *lengths* of the sides of ΔABC.

We now relate the notation we have just introduced to the Pythagorean Theorem. Suppose that ΔABC is a triangle that has a right angle at the vertex C. In other words, the two sides $\overline{AC}$ and $\overline{BC}$ form a right angle. Then the Pythagorean Theorem tells us that

$$(AC)^2 + (BC)^2 = (AB)^2.$$



More about Similar Triangles

We now present a more detailed discussion of similar triangles.

Recall: two triangles T and T´ are *similar* if there is a correspondence between the sides of T and the sides of T´ and there is a real number r > 0 called a *scale factor* such that the length of each side of T´ is equal to r times the length of the corresponding side of T.

Let ΔABC and $\Delta A'B'C'$ be triangles. We write $\Delta ABC \sim \Delta A'B'C'$ if there is a real number $r > 0$ such that

$$A'B' = r(AB), \quad A'C' = r(AC) \quad \text{and} \quad B'C' = r(BC).$$

Thus, the relationship $\Delta ABC \sim \Delta A'B'C'$ implies that the two triangles ΔABC and $\Delta A'B'C'$ are similar. However, the relationship $\Delta ABC \sim \Delta A'B'C'$ conveys more than just the fact that the two triangles ΔABC and $\Delta A'B'C'$ are similar. It also includes the information that the correspondence inherent in the similarity relation between ΔABC and $\Delta A'B'C'$ associates:

- side $\overline{AB}$ of ΔABC with side $\overline{A'B'}$ of $\Delta A'B'C'$,
- side $\overline{AC}$ of ΔABC with side $\overline{A'C'}$ of $\Delta A'B'C'$ and
- side $\overline{BC}$ of ΔABC with side $\overline{B'C'}$ of $\Delta A'B'C'$.

We will now state several fundamental and useful facts about similar triangles. These facts are all *provable,* and hence are called *theorems.* (A *theorem* is a *provable fact.*)

Before starting these theorems, we recall some logical terminology. Let P and Q be statements. Then the *implication* "if P, then Q" can also be expressed as "P implies Q". The implication "Q implies P" is called the *converse* of the implication "P implies Q". Recall that an implication may be true while its converse is false, and vice versa. However, the statement "P if and only if Q" means that both the implication "P implies Q" and its converse "Q implies P" are true. In other words, "P if and only if Q" means "P implies Q, and Q implies P". Another way to express "P if and only if Q" is to say "the statements P and Q are *equivalent*". Thus, "P and Q are equivalent" means "P implies Q and Q implies P". Consequently, saying "the statements P, Q, and R are equivalent" means "(i) P implies Q, (ii) Q implies P, (iii) Q implies R, (iv) R implies Q, (v) P implies R, and (vi) R implies P".

Notice that implication is *transitive*; in other words, the implication "P implies R" automatically follows from the two implications "P implies Q" and "Q implies R". Hence, if you were trying to prove that statements P, Q, and R are equivalent it would not be necessary to prove all of the implication (i) through (vi) listed above. It would be sufficient to prove implications (i) through (iv) only, because implication (v) follows automatically from implications (i) and (iii), and implication (vi) follows automatically from implication (ii) and (iv).

We now state the first theorem about similar triangles. The theorem reveals that we can use simple algebra to recast the definition of "similarity" so that the scale factor is not explicitly mentioned.

The *Side Theorem* for Similar Triangles. Suppose ΔABC and $\Delta A'B'C'$ are triangles. Then the following statements are equivalent.

a) $\Delta ABC \sim \Delta A'B'C'$.

b) $\dfrac{A'B'}{AB} = \dfrac{A'C'}{AC} = \dfrac{B'C'}{BC}$.

c) $\dfrac{AB}{AC} = \dfrac{A'B'}{A'C'}$ and $\dfrac{AB}{BC} = \dfrac{A'B'}{B'C'}$.

Proof. First we prove "a) implies b)". Assume that a) is true. Then $\Delta ABC \sim \Delta A'B'C'$. Hence there is a scale factor $r > 0$ such that

$$A'B' = r(AB),\ \ A'C' = r(AC),\ \text{ and }\ B'C' = r(BC).$$

Hence

$$\frac{A'B'}{AB} = r,\ \ \frac{A'C'}{AC} = r,\ \text{ and }\ \frac{B'C'}{BC} = r.$$

Therefore

$$\frac{A'B'}{AB} = \frac{A'C'}{AC} = \frac{B'C'}{BC}.$$

Thus b) is true. We have now proved that a) implies b).

Second we prove that b) implies a). Assume that b) is true. Then

$$\frac{A'B'}{AB} = \frac{A'C'}{AC} = \frac{B'C'}{BC}.$$

Let $r = \dfrac{A'B'}{AB}$. Then $r = \dfrac{A'C'}{AC}$ and $r = \dfrac{B'C'}{BC}$. Therefore,

$$A'B' = r(AB),\ \ A'C' = r(AC)\ \text{ and }\ B'C' = r(BC).$$

Thus r is a scale factor relating the lengths of the sides of ΔABC to the lengths of the sides of $\Delta A'B'C'$. So $\Delta ABC \sim \Delta A'B'C'$. Hence a) is true. We have proved b) implies a).

Since a) implies b) and b) implies a), then statements a) and b) are equivalent.

Third we prove b) implies c). Assume b) is true. Then

$$\frac{A'B'}{AB} = \frac{A'C'}{AC} = \frac{B'C'}{BC}.$$

Hence

$$\frac{A'B'}{AB} = \frac{A'C'}{AC} \quad \text{and} \quad \frac{A'B'}{AB} = \frac{B'C'}{BC}$$

A little algebra converts the first equation to $\frac{AB}{AC} = \frac{A'B'}{A'C'}$ and the second equation to $\frac{AB}{BC} = \frac{A'B'}{B'C'}$. Thus c) is true. We have proved b) implies c).

Fourth we prove c) implies b). Assume c) is true. Then

$$\frac{AB}{AC} = \frac{A'B'}{A'C'} \quad \text{and} \quad \frac{AB}{BC} = \frac{A'B'}{B'C'}.$$

A little algebra converts the first equation to $\frac{A'B'}{AB} = \frac{A'C'}{AC}$ and the second equation to $\frac{A'B'}{AB} = \frac{B'C'}{BC}$. We can combine these two equations into one:

$$\frac{A'B'}{AB} = \frac{A'C'}{AC} = \frac{B'C'}{BC}.$$

Hence, b) is true. We have proved c) implies b).

Since b) implies c) and c) implies b), then statements b) and c) are equivalent.

Finally, the implication "a) implies c)" follows by transitivity from the already proved implications "a) implies b)" and "b) implies c)". Similarly, the implication "c) implies a)" follows by transitivity from the already proved implications "c) implies b)" and "b) implies a)". Therefore, since "a) implies c)" and "c) implies a)" are true, then statements a) and c) are equivalent.

We have shown that statements a), b), and c) are equivalent. This finishes the proof of the Side Theorem for Similar Triangles.

What is the significance of the Side Theorem for Similar Triangles? Statement b) shows us that it is possible to express the fact that two triangles are similar without explicitly mentioning the scale factor. Statement c) also expresses triangle similarities without using a scale factor. However, statement c) has another important use.

Notice that in each ratio appearing in statement c) the numerators and denominators are side lengths of the *same* triangle. These ratios never mix side lengths of different triangles. As a consequence, if the side lengths of ΔABC are expressed on terms of one unit (say miles) while the side lengths of $\Delta A'B'C'$ are expressed in terms of a different unit (say inches), then statement c) can be applied *without converting these lengths to a common unit*. Because the lengths appearing in each ratio are expressed in the same units, the units in each ratio essentially “cancel out” yielding ratios that are unitless “pure” numbers. When you use statement c) to decide whether two triangles are similar, you simply check that the equations in statement c) between the unitless pure number ratios hold without bothering to convert the side lengths of the two triangles to a common unit.

Here is a example of an application of statement c) of the Side Theorem for Similar Triangles. Suppose ΔABC is a triangle with side lengths

$$AB = 6 \text{ in}, \quad BC = 9 \text{ in}, \quad AC = 12 \text{ in}$$

and suppose ΔA´B´C´ is a triangle with side lengths

$$A'B' = 10 \text{ mi}, \quad B'C' = 15 \text{ mi}, \quad A'C' = 20 \text{ mi}.$$

Then

$$\frac{AB}{AC} = \frac{6}{12} = \frac{1}{2} = \frac{10}{20} = \frac{A'B'}{A'C'} \quad \text{and} \quad \frac{AB}{BC} = \frac{6}{9} = \frac{2}{3} = \frac{10}{15} = \frac{A'B'}{B'C'}.$$

Now Similar Triangle Theorem 1 implies that ΔABC ~ ΔA´B´C´. The crucial point here is that we established the similarity relationship between ΔABC and ΔA´B´C´ without converting the side lengths of these two triangles to same units.

We remark that in everyday life, similar triangles commonly occur with the sides of one of the triangles measured in inches and the sides of the other triangle measured in miles. If on a map that has a scale of 1 inch = $1^2/_3$ miles, you draw a triangle with side lengths of 6 in, 9 in and 12 in, then this triangle represents a similar triangle with side lengths 10 mi, 15 mi and 20 mi.

Our second theorem about similar triangles connects the similarity relation between triangles to the relation between their angles. This theorem is one of the fundamental results of plane or Euclidean geometry. We will state this theorem without proof. (The proof is a bit complicated and would divert us too far from the main path of the course.)

The *Angle Theorem* for Similar Triangles. Suppose ΔABC and ΔA´B´C´ are triangles.

a) If ΔABC ~ ΔA´B´C´, then

- the angle of ΔABC at A has the same measure as the angle of ΔA´B´C´ at A´,
- the angle of ΔABC at B has the same measure as the angle of ΔA´B´C´ at B´, and
- the angle of ΔABC at C has the same measure as the angle of ΔA´B´C´ at C´.

b) If

- the angle of ΔABC at A has the same measure as the angle of ΔA´B´C´ at A´, and
- the angle of ΔABC at B has the same measure as the angle of ΔA´B´C´ at B´,

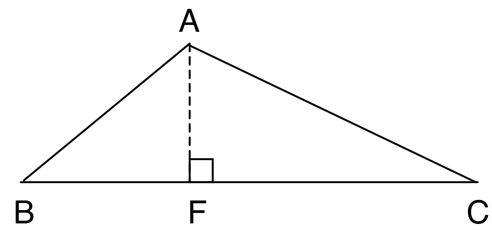
then ΔABC ~ ΔA´B´C´.

Statement a) of the Angle Theorem for Similar Triangles tells us that if two triangles are similar, then the angles of one triangle have the same size as the corresponding angles of the other triangle. Observe that Statement b) of the Angle Theorem for Similar Triangles reverses the logical order of this statement. In other words, Statement b) is a sort of *converse* to Statement a). Statement b) says that if *two* or the three angles of one triangle have the same size as *two* of the three angles of a second triangle, then the two triangles are similar.

It is a little misleading to stress the fact that statement b) requires knowledge of the sizes of only *two* of the three angles of each triangle. The reason is that if we know the sizes of two of the three angles of a triangle, then we also know the size of the third angle. **Why?**

We will see that Statement b) of the Angle Theorem for Similar Triangles is a very useful fact. This is because we will often be confronted with two triangles for which we know that two angles of the first triangle are the same size as two angles of the second triangle, and we will want to conclude that the two triangles are similar. Statement b) tells us that this conclusion is justified.

We will state one more useful theorem about similar triangles. First we recall another well known geometry term. An *altitude* of a triangle T is a line segment that has one endpoint at a vertex of T and has the other endpoint of the opposite side of T and is perpendicular to that side of T.



In this picture, $\overline{AF}$ is an altitude of ΔABC.

The following theorem says that if two triangles T and T´ are similar, then the scale factor which relates the lengths of corresponding sides of T and T´ also relates the lengths of corresponding altitudes of T and T´. This theorem can also be formulated in terms of ratios between side lengths in a way that doesn't explicitly mention scale factors. Furthermore, in this formulation there are two different ways to express the ratios between side lengths. One can either form the ratio between the length of a side of T and the length of the corresponding side of T´, or one can form ratios in which the numerators and denominators are lengths of sides that come from a single triangle. (This is the kind of distinction that differentiates statements b) and c) in the Side Theorem for Similar Triangles.)

The *Altitude Theorem* for Similar Triangles. Suppose ΔABC and $\Delta A'B'C'$ are triangles such that $\Delta ABC \sim \Delta A'B'C'$, and suppose $\overline{AF}$ is an altitude of ΔABC and $\overline{A'F'}$ is an altitude of $\Delta A'B'C'$.

a) If r is the scale factor relating the side lengths of ΔABC to the side lengths of $\Delta A'B'C'$ (so that $r(AB) = A'B'$, $r(AC) = A'C'$, and $r(BC) = B'C'$), then $r(AF) = A'F'$.

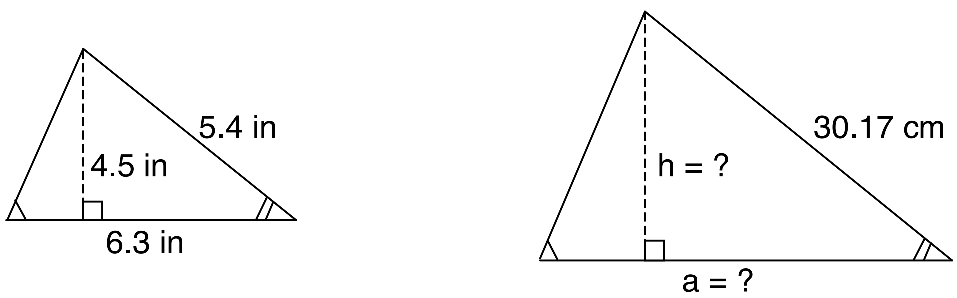
b) $\dfrac{A'F'}{AF} = \dfrac{A'B'}{AB} = \dfrac{A'C'}{AC} = \dfrac{B'C'}{BC}$.

c) $\dfrac{A'F'}{A'B'} = \dfrac{AF}{AB}$, $\dfrac{A'F'}{A'C'} = \dfrac{AF}{AC}$, and $\dfrac{A'F'}{B'C'} = \dfrac{AF}{BC}$.

Simple algebraic manipulations like those used in the proof of the Side Theorem for Similar Triangles can be used to convert any one of the statements a), b), and c) into any other. We assign the proof of this theorem as Homework Problem 4.

Statement c) of the Altitude Theorem for Similar Triangles can be applied even if the side lengths of one of the two triangles are measured in different units than the sides lengths of the other triangle. This is because in Statement c), the side lengths appearing in the numerator and denominator of each ratio always come from the same triangle and never mix units.

Here is another example of an application of the three theorems for similar triangles. Consider the two triangles shown below. We must find the lengths a and h of the indicated line segments. The Angle Theorem for Similar Triangles tells us that the



two triangles are similar because two of the angles of one of the triangles have the same measures as two of the angles of the other triangle. Statement c) of the Side Theorem for Similar Triangles implies that

$$\frac{a}{30.17} = \frac{6.3}{5.4},$$

and Statement c) of the Altitude Theorem for Similar Triangles implies that

$$\frac{h}{30.17} = \frac{4.5}{5.4}.$$

Before proceeding further with these calculations, observe that this is an appropriate situation in which to use *significant figures,* because it involves division and the input data is given in different units to different degrees of accuracy. Multiplying on both sides of these equations by 30.17 gives:

$$a = \left(\frac{6.3}{5.4}\right)(30.17) \quad \text{and} \quad h = \left(\frac{4.5}{5.4}\right)(30.17).$$

Now, using a pocket calculator, we get:

$$a = (1.16666\ldots)(30.17) = 35.1983333\ldots$$

and

$$h = (.83333\ldots)(30.17) = 25.1416666\ldots$$

The triangle which contains the segments of length a and h also contains a segment of length 30.17 cm to the nearest hundredth of a centimeter. If we round the values for a and h to the nearest hundredth of a centimeter, we get:

$$a = 35.20 \text{ cm} \quad \text{and} \quad h = 25.14 \text{ cm}.$$

These values of a and h have 4 significant figures. However, the lengths of the sides and altitude of the other triangle have only 2 significant figures. So expressing a and h to 4 significant figures is unjustifiable in this situation. The values for a and h are only reliable up to 2 significant figures. Hence, we must round the values of a and h to 2 significant figures, giving us:

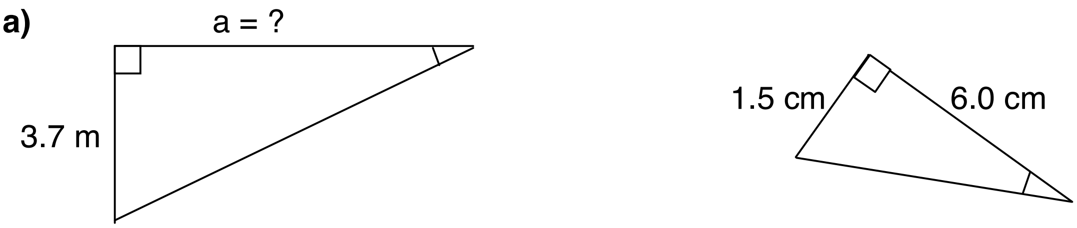
$$a = 35 \text{ cm} \quad \text{and} \quad h = 25 \text{ cm}.$$

Note that we obey the *Calculation Rules* while performing this calculation.

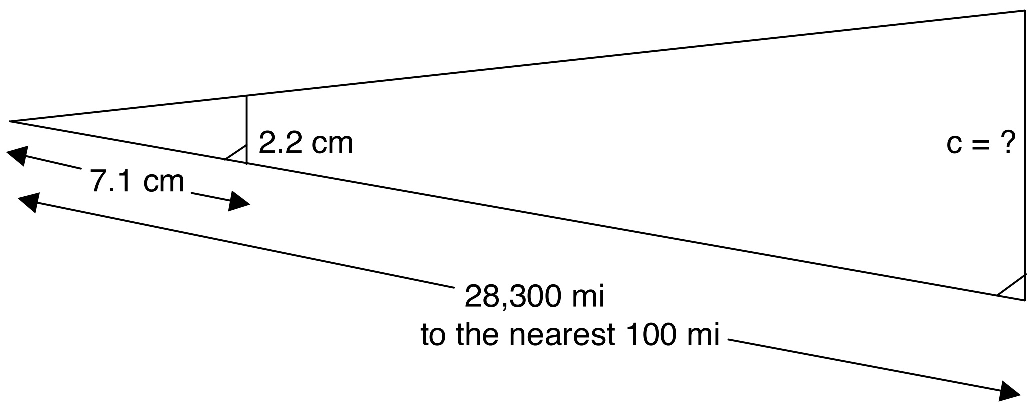
Activity 4. Groups should read and discuss Homework Problem 5 (below) from this lesson and report their results to the class.

Homework Problem 1. In each of parts a) through g) of this problem, justify why the two triangles are similar by citing one of the three theorems for similar triangles. Then find the length of the unknown line segment. Be sure to obey the two *Calculation Rules*.

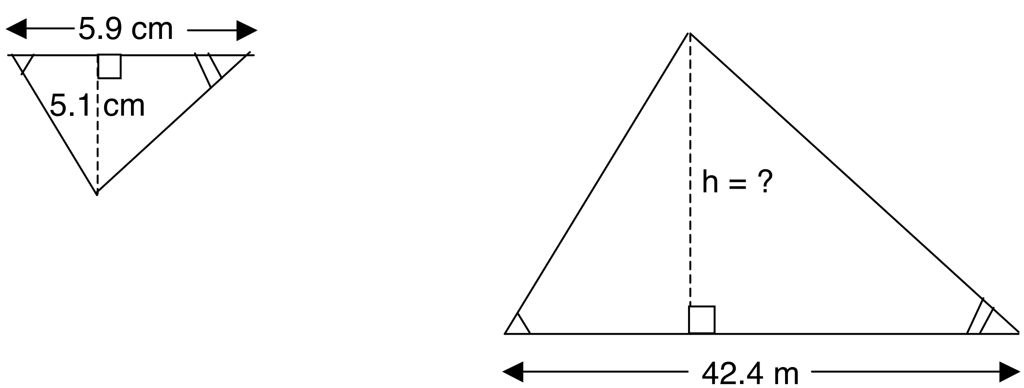
a)



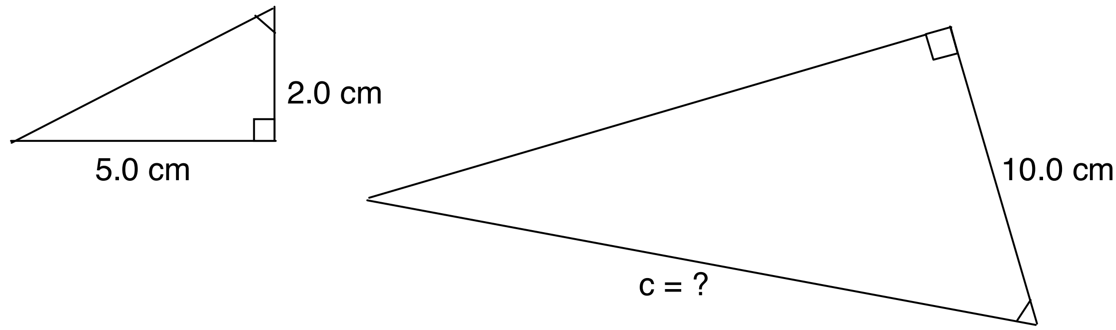
b)



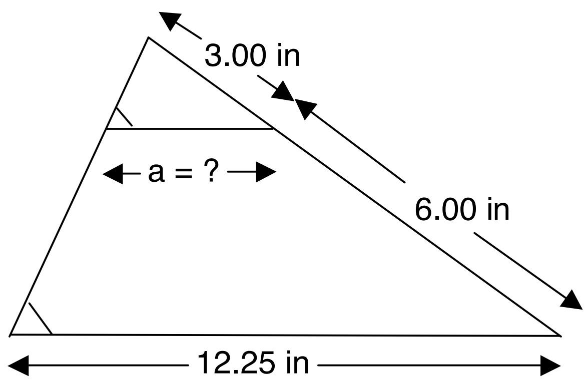
c)



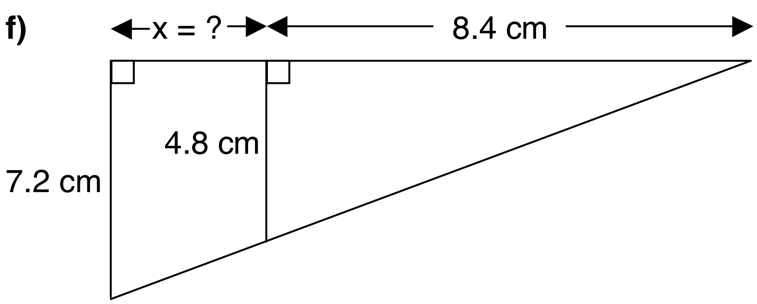
d)



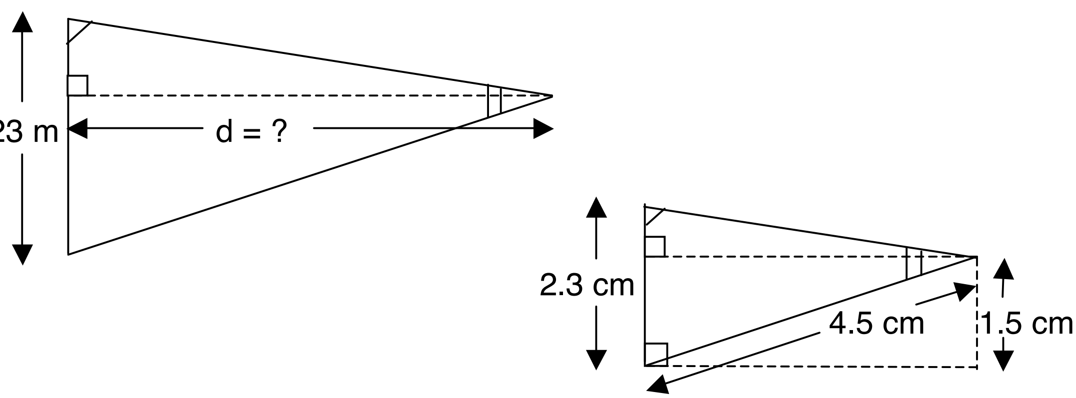
e)



f)



g)



Homework Problem 2. Flagpole Number One is 57 feet tall, Flagpole Number Two is 84 feet tall, and the distance between them is 283 feet (all measured to the nearest foot). If you stand at a point where from your perspective, the top of Flagpole Number Two is directly behind the top of Flagpole Number One, how far are you standing from Flagpole Number One?

Homework Problem 3. Standing on top of a hill, you see a car driving along a distant road. If you hold your arm straight out in front of you and stick up your thumb, then from your perspective, the width of your thumb exactly covers the length of the car. Suppose the distance from your eye to your thumb is .61 meters and your thumb is 2.5 centimeters wide. Estimate the distance between you and the car. (Here's a helpful piece of information: the typical car is about 5.0 meters long.)

Homework Problem 4. It is possible to write a proof of the Altitude Theorem for Similar Triangles that appeals to the Side and Angle Theorems for Similar Triangles. Write such a proof. Here are three suggestions.

a) In the first part of the proof, assume that $\Delta ABC \sim \Delta A'B'C'$ and that $\overline{AF}$ and $\overline{A'F'}$ are altitudes of ΔABC and $\Delta A'B'C'$, respectively. Appeal to the Angle Theorem for Similar Triangles to prove that $\Delta ABF \sim \Delta A'B'F'$.

b) Next use the Side Theorem for Similar Triangles to prove statement b) of the Altitude Theorem for Similar Triangles.

c) Finally use algebra to prove statement c) of the Altitude Theorem for Similar Triangles from statement b) of this theorem.

Homework Problem 5: Melissa's Problem. Melissa wants to measure the height of the flagpole in front of her school. She stands at a point that is 26 m from the flagpole and holds a meter stick vertically so that, from her perspective, the meter stick lines up next to the flagpole with the bottom (zero end) of the meter stick lining up next to the bottom of the flagpole. (This does not mean that the bottom of the meter stick is on the ground.) The top of the flagpole lines up next to the 37 cm mark on the meter stick. Also, Melissa's eyes are 158 cm above the ground, and the distance from Melissa's eyes to the bottom of the meter stick is 44 cm. Melissa uses all this information to calculate the height of the flag pole. What answer do you think she should get?

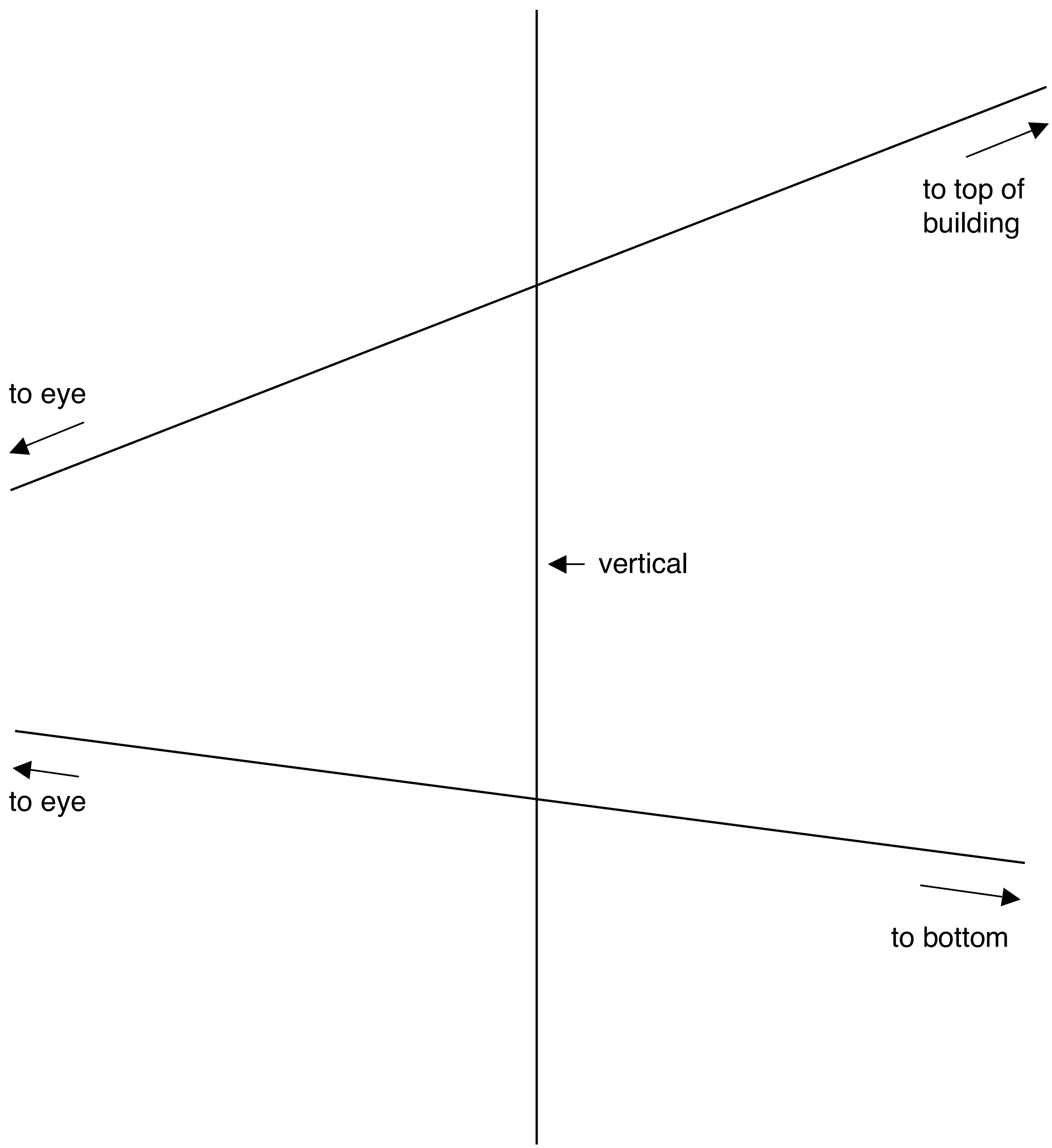
Hint: First draw a picture of the situation described in the problem. Or else try to make a scale model of this situation using *string* to represent the path light travels from a point on the flagpole through a point on the ruler to Melissa's eye. As a second step, find the distance from Melissa's eyes to the bottom of the flagpole.

Homework Problem 6: Bernardo's Problem. Bernardo knows that the flagpole in front of his school is 14 meters high. He wants to measure the distance from the front door of the school building to the flagpole. So he stands next to the front door and holds a meter stick vertically so that, from his perspective, the meter stick lines up next to the flagpole with the bottom of the meter stick lining up next to the bottom of the flagpole. The top of the flagpole lines up next to the 34 cm mark on the meter stick. Also, Bernardo's eyes are 163 cm above the ground, and the distance from Bernardo's eyes to the bottom of the meter stick is 46 cm. Bernardo uses all this information to calculate the distance from the front door of the school building to the flagpole. What answer do you think he should get?

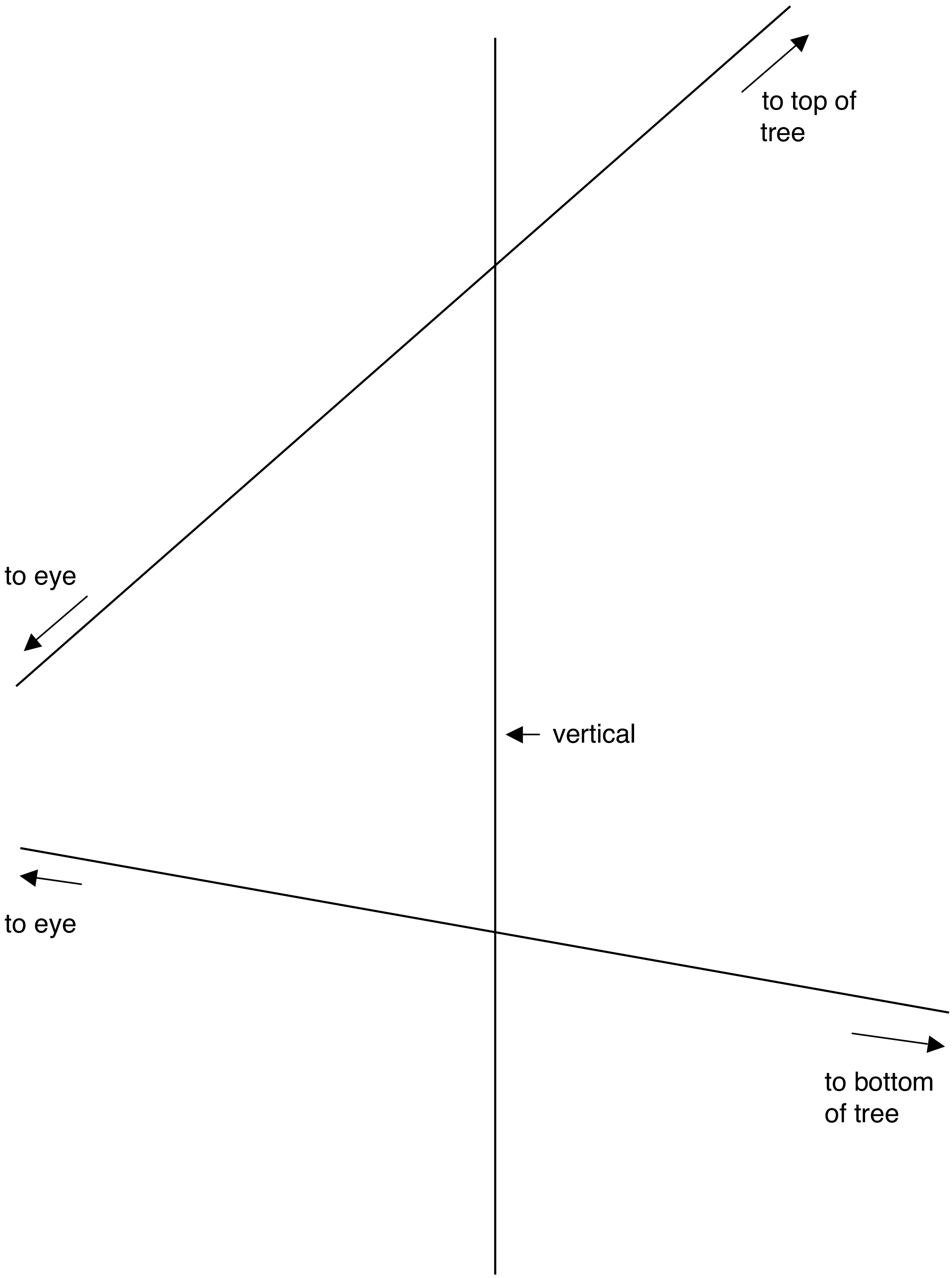
Hint: First draw a picture of the situation described in the problem, or make a scale model using string to represent light paths.

Homework Problem 7: Sharonda's Problem. Sharonda wants to measure the height of the UWM Education Building Enderis Hall. So she stands at a point that is 70 m from a corner of the building and holds a clipboard with a piece of paper on it vertically so that she can draw lines on the paper that run from her eye to the corner of the building. She draws three lines on the paper: one from her eye to the bottom of the building, one from her eye to the top of the building, and one vertical line. These lines are shown on the next page. Next she tapes a piece of paper to the left edge of her diagram and extends the two non-vertical lines until they meet, forming a triangle. She then measures some distances in this extended diagram on the two pieces of paper and calculates the height of the building. What answer do you think she should get?

Hint: Tape a piece of paper to the left edge of this paper, complete the triangle and measure the appropriate distances.



Homework Problem 8: Kristin's Problem. Kristin wants to measure the distance from the front door of her house to a 73-foot tall willow tree in her front yard. She stands at her front door and holds a clipboard with a piece of paper on it vertically so that she can draw lines on the paper that run from her eye to the tree. She draws three lines on the paper: one form her eye to the bottom of the tree, one from her eye to the top of the tree, and one vertical line. Her diagram is shown on the next page. Next she extends the two non-vertical lines until they meet (taping another piece of paper to the left edge of her diagram if needed). She then constructs another line in her extended diagram and measures some distances in the diagram. Finally, she uses these measurements to calculate the distance from her front door to the tree. What answer do you think she should get? Is there any danger that if the tree blew over in a storm that it would hit her front door?


to top of
tree
to eye
vertical
to eye
to bottom
of tree