## Lesson 3: The Converse of the Pythagorean Theorem, Pythagorean Triples and the 3-Dimensional Pythagorean Theorem

Recall that the Pythagorean Theorem tells us how to calculate the length of one side of a right triangle if we know the lengths of the other two sides. We restate it:

**The Pythagorean Theorem.** In a triangle which has sides of length a, b and c, if the sides of length a and b form a right angle, then $a^2 + b^2 = c^2$.

In the previous lesson we looked at some (hopefully) convincing mathematical arguments for the Pythagorean Theorem. Such arguments are called *proofs.* The proofs we saw in Lesson 2 all depend on the idea of area. In addition to these proofs, there are a wide variety of proofs that depend on fundamental geometric concepts other than area; however, the study of such proofs is outside the scope of this course.

Let us examine the logical structure of the statement of the Pythagorean Theorem. Notice that the essential part of the Pythagorean Theorem is a statement of the form

"If _____, then _____"

where the blanks are filled in by simpler statements. A statement of this form is often called an *implication* or a *conditional statement,* and most mathematical statements can be put into this form.

Here are some other examples of statements which are implications.

*If it is raining, then it is cloudy.*

*If an integer (whole number) n is greater than 4.5, then n is greater than 4.*

In general, implications are *compound* statements of the form

"If P, then Q"

where P and Q are themselves simpler statements. The implication "if P, then Q" can be expressed in other ways. Here are some of the possibilities:

"If P is true, then Q is true",

"Whenever P is true, then Q must be true",

"P implies Q".

If we start with an implication "If P, then Q", then we can form a new implication from it by reversing its logical order. The new implication would be of the form

"If Q, then P".

The new implication “if Q, then P” is referred to as the *converse* of the original implication “if P, then Q”.

Here are the converses of the two implications stated above.

*If it is cloudy, then it is raining.*

*If an integer n is greater than 4, then n is greater than 4.5.*

Here is a fundamental question which frequently arises in mathematics:

If an implication is true, must its converse be true?

The answer is: sometimes *no* and sometimes *yes*. It depends on the implication.

**Activity 1.** The groups should consider the questions a) through e) and report their answers to the class. Then the entire class should discuss problem f).

**a)** Is the implication “If today is Wednesday, then tomorrow is Thursday” true? Is its converse true?

**b)** Is the implication “If a figure F is a square, then F is a rectangle” true? Is its converse true?

**c)** Is the implication “If an integer n is greater than 4.5, then n is greater than 4” true? Is its converse true?

**d)** Is the implication “If a real number[1] x is greater than 4.5, then x is greater than 4” true? Is its converse true?

**e)** Is the implication “If a quadrilateral Q has four sides of equal length, then Q is a square” true? Is its converse true?

**f)** State the converse of the Pythagorean Theorem by completing the following sentence. “In a triangle which has sides of length a, b and c, …” Write your answer below.

**The Converse of the Pythagorean Theorem.** In a triangle which has sides of length a, b and c,

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[1] The *real numbers* include all the integers – 0, ±1, ±2, ±3, … –, all the fractions (or *rational numbers*) like ±1/2, ± 13/18, ±763/13, etc., and all the *irrational numbers* like $\pm\sqrt{2}$, $\pm\sqrt{17}$ and ±π.

The problem in Activity 1 f) naturally leads to the following question:

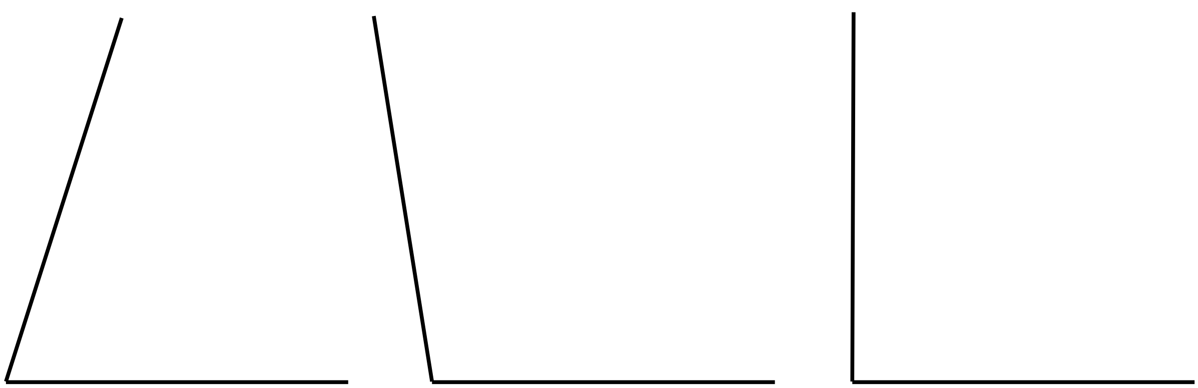
Is the converse of the Pythagorean Theorem true?

If the converse of the Pythagorean Theorem is true, then it gives us a tool for constructing right angles and for testing whether a given angle is truly a right angle. Indeed, to construct a right angle, we simply need to choose positive numbers a, b and c satisfying $a^2 + b^2 = c^2$ and construct a triangle with sides of length a, b and c. Then the converse of the Pythagorean Theorem tells us that the angle between the sides of length a and b must be a right angle.

Now we consider the question: Is the converse of the Pythagorean Theorem true? In the following two activities, we will test the truth of the converse of the Pythagorean Theorem *experimentally*.

**Activity 2.** In this activity you are asked to develop your ability to perform two geometric tasks. First you are asked to devise a method for recognizing right angles. Second you are asked to devise a method for constructing triangles with given side lengths.

**a)** Each student should copy the three angles below on patty paper and devise a method for testing whether these angles are right angles by folding the patty paper. Students should then report their results to the class.



**b)** Each group should devise a method for constructing one of the triangles of side lengths a, b and c listed below using a ruler and patty paper or a compass.

**i)** $a = 3^1/_4$ in, $b = 2^3/_4$ in, $c = 1^1/_4$ in

**ii)** a = 7.2 cm, b = 9.0 cm, c = 3.1 cm

**iii)** $a = 2^3/_4$ in, $b = 3^1/_4$ in, $c = 3^1/_2$ in

Groups should report their results of their activities to the class.

**Activity 3.** This activity is intended to provide gather “experimental evidence” concerning the truth of the converse of the Pythagorean Theorem. Here are seven triples of numbers a, b, c that satisfy the equation $a^2 + b^2 = c^2$.

**1)** a = 5 cm, b = 12 cm, c = 13 cm

**2)** a = $1^1/_2$ in, b = 2 in, c = $2^1/_2$ in

**3)** a = 12 cm, b = 6.4 cm, c = 13.6 cm

**4)** a = $1^1/_2$ in, b = $4^3/_8$ in, c = $4^5/_8$ in

**5)** a = 10 cm, b = 10.5 cm, c = 14.5 cm

**6)** a = $3^{15}/_{16}$ in, b = 1 in, c = $4^1/_{16}$ in

**7)** a = 9 cm, b = 5.6 cm, c = 10.6 cm

Different triples should be assigned to different groups.

- First each group should verify that its triple satisfies the equation $a^2 + b^2 = c^2$.
- Then each group should construct a triangle on a piece of paper whose side lengths are the numbers a, b and c from the triple assigned to the group. Use one of the methods devised in Activity 2 b). *(Don’t construct a right angle first!)*
- If the converse of the Pythagorean Theorem is true, then these triangles should have right angles between the sides of length a and b. Each group should test whether the angle of its triangle between the sides of length a and b is actually a right angle. Use one of the methods devised in Activity 2 a).
- Each group should report the results of its efforts to the class.

Activity 3 is intended to convince you of the following fact:

*The converse of the Pythagorean Theorem is **true**.*

Thus, if a, b and c are any positive numbers such that $a^2 + b^2 = c^2$, then any triangle with side lengths a, b and c will have a right angle between the sides of length a and b.

Note that

$$3^2 + 4^2 = 5^2.$$

$$(\text{Proof: } 3^2 + 4^2 = 9 + 16 = 25 = 5^2.)$$

Therefore, the converse of the Pythagorean Theorem implies that any triangle with sides of length 3 cm, 4 cm and 5 cm is a right triangle such that the angle between the sides of length 3 cm and 4 cm is a right angle.

Suppose that a, b and c are positive numbers such that $a^2 + b^2 = c^2$. Note that if we multiply a, b and c by the same positive number x, then the equation

$$(xa)^2 + (xb)^2 = (xc)^2$$

is satisfied.

$$\text{(Proof: } (xa)^2 + (xb)^2 = x^2a^2 + x^2b^2 = x^2(a^2 + b^2) = x^2c^2 = (xc)^2.\text{)}$$

It follows that if a, b and c are positive numbers such that $a^2 + b^2 = c^2$, then for each positive number x, every triangle with sides of length xa, xb and xc is a right triangle with a right angle between the sides of length xa and xb. In particular, since

$$3^2 + 4^2 = 5^2,$$

then not only is every triangle with sides of length

3 cm, 4 cm and 5 cm

a right triangle. Also every triangle with sides of length

$2\times3 = 6$ cm, $2\times4 = 8$ cm and $2\times5 = 10$ cm

is a right triangle. Also every triangle with sides of length

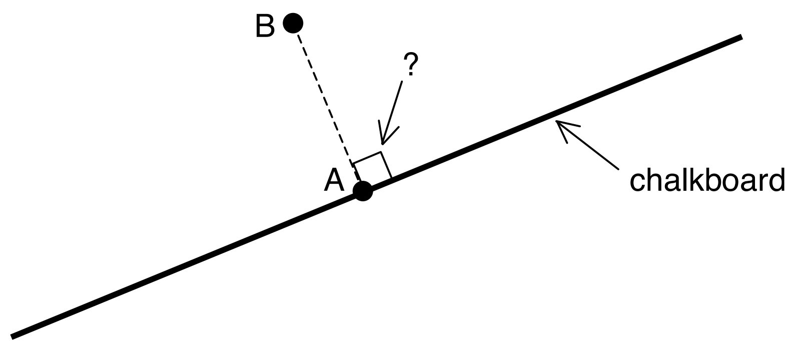
$(^1/_4)\times3 = .75$ cm, $(^1/_4)\times4 = 1$ cm, $(^1/_4)\times5 = 1.25$ cm

is a right triangle.

**Activity 4.** The class should discuss the following two questions.

**a)** A seven foot piece of string has a knot tied at a point that is 3 feet from one end of the string and 4 feet from the other end. Three members of the class whom we will call "A", "B" and "C" hold the string at different points. A and C hold the two ends of the string while B holds the string at the knot. A, B and C back away from each other until the string is taut, and they *don't* stand in a straight line. Therefore, the string makes an angle at the knot where B holds it. How can we decide whether the angle made by the string is less than, equal to, or greater than 90º?

**b)** Two class members whom we will call "A" and "B" hold the ends of an 8 foot string at the same height. Also A holds his end of the string at a mark on the classroom chalkboard. How can we move B without moving A so that the string they hold is perpendicular to the chalkboard? (We are allowed to use more string, a third member of the class (named "C") and converse of the Pythagorean Theorem.) (See the figure at the top of the next page.)




**Activity 5.** The groups will discuss the following questions and report their results to the class. Suppose a point P and a line L are drawn on a sheet of paper such that P does not lie on L.

**a)** What does the *distance form P to L* mean?

**b)** How can you *drop a perpendicular from P to L*?

**c)** How can you measure the distance from P to L?

## Pythagorean Triples

If a, b and c are positive integers (whole numbers) such that $a^2 + b^2 = c^2$, then the three numbers a, b and c are called a *Pythagorean triple*. The Converse of the Pythagorean Theorem tells us that if the side lengths of a triangle form a Pythagorean triple, then the triangle is a right triangle.

We have seen that 3, 4, 5 and 6, 8, 10 are Pythagorean triples. Thus, any triangle with side lengths 3 cm, 4 cm, 5 cm is a right triangle. Similarly, any triangle with side lengths 6 in, 8 in, 10 in is a right triangle,

Observe that 5, 12, 13 is another Pythagorean triple.

$$\text{(Proof: } 5^2 + 12^2 = 25 + 144 = 169 = 13^2.)$$

Therefore, any triangle with side lengths 5 m, 12 m, 13 m is a right triangle.

**Further information about Pythagorean Triples.** Here is a method for producing Pythagorean triples. Let u and v be any two positive integers such that $u < v$. Next let

$$a = v^2 - u^2, \qquad b = 2vu \qquad \text{and} \qquad c = v^2 + u^2.$$

Then the three numbers a, b, c form a Pythagorean triple.

For example, if $u = 1$ and $v = 2$, then

$$a = 2^2 - 1^2 = 4 - 1 = 3, \qquad b = 2x2x1 = 4, \qquad \text{and} \qquad c = 2^2 + 1^2 = 4 + 1 = 5.$$

Similarly, if $u = 2$ and $v = 3$, then

$$a = 3^2 - 2^2 = 9 - 4 = 5, \qquad b = 2x2x3 = 12, \qquad \text{and} \qquad c = 3^2 + 2^2 = 9 + 4 = 13.$$

We note that the Pythagorean triple 3, 4, 5 has no common factor. Hence, it is not a multiple of a smaller Pythagorean triple. We call such a Pythagorean triple a *prime Pythagorean triple.* Similarly, 5, 12, 13 is a prime Pythagorean triple. However, 6, 8, 10 is not a prime Pythagorean triple because is it a multiple of 3, 4, 5.

The method described above for producing Pythagorean triples will produce *prime* Pythagorean triples if the positive integers u and v have two additional properties:

- u and v have no common factor, and
- u and v are not both odd.

On the next page is a table of prime Pythagorean triples which is produced by the method just described. We remark that every possible prime Pythagorean triple is produced by this method and would eventually appear in this table if we were to extend the table far enough.

**Table of Prime Pythagorean Triples**

| u | v | a | b | c |
|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 |
| 1 | 4 | 15 | 8 | 17 |
| 1 | 6 | 35 | 12 | 37 |
| 1 | 8 | 63 | 16 | 65 |
| 1 | 10 | 99 | 20 | 101 |
| 2 | 3 | 5 | 12 | 13 |
| 2 | 5 | 21 | 20 | 29 |
| 2 | 7 | 45 | 28 | 53 |
| 2 | 9 | 77 | 36 | 85 |
| 2 | 11 | 117 | 44 | 125 |
| 3 | 4 | 7 | 24 | 25 |
| 3 | 8 | 55 | 48 | 73 |
| 3 | 10 | 91 | 60 | 109 |
| 3 | 14 | 187 | 84 | 205 |
| 3 | 16 | 247 | 96 | 265 |
| 4 | 5 | 9 | 40 | 41 |
| 4 | 7 | 33 | 56 | 65 |
| 4 | 9 | 65 | 72 | 97 |
| 4 | 11 | 105 | 88 | 137 |
| 4 | 13 | 153 | 104 | 185 |
| 5 | 6 | 11 | 60 | 61 |
| 5 | 8 | 39 | 80 | 89 |
| 5 | 12 | 119 | 120 | 169 |
| 5 | 14 | 171 | 140 | 221 |
| 5 | 16 | 231 | 160 | 281 |
| 6 | 7 | 13 | 84 | 85 |
| 6 | 11 | 85 | 132 | 157 |
| 6 | 13 | 133 | 156 | 205 |
| 6 | 17 | 253 | 204 | 325 |
| 6 | 19 | 325 | 228 | 397 |

We verify that the triple 325, 228, 397 appearing in the bottom line of this table is truly a Pythagorean triple:

$$325^2 + 228^2 = 105{,}625 + 51{,}984 = 157{,}609 = 397^2.$$

(Here are two examples illustrating that if we want to produce *prime* Pythagorean triples, then the positive integers u < v must have the two additional properties:

- u and v have no common factor, and
- u and v are not both odd.

Without these two additional properties, the Pythagorean triples produced from u and v will not be prime. For example:

Here is one example. If u = 2 and v = 4, then a = $4^2 - 2^2$ = 12 = 4x3, b = 2x4x2 = 16 = 4x4, and c = $4^2 + 2^2$ = 20 = 4x5. Thus, if u and v have a common factor of 2, then a, b and c have a common factor of 4.

Here is another example. If u = 1 and v = 3, then a = $3^2 - 1^2$ = 8 = 2x4, b = 2x3x1 = 6 = 2x3, and c = $3^2 + 1^2$ = 10 = 2x5. Thus, if u and v are both odd, then a, b and c have a common factor of 2.

## ***If-and-only-if*** **Statements**

We briefly return to logic. Recall that an *implication* is a statement of the form "if P, then Q" where P and Q are themselves statements. Furthermore, the *converse* of this implication is the statement "if Q, then P". Recall that if an implication is true, its converse might be true or false. When both the implication "if P, then Q" and its converse "if Q, then P" are true, then we can express this by saying "P *if and only if* Q". Thus, an *if-and-only-if* statement is true precisely when both the associated implication and its converse are true. In other words, "P if and only if Q" means "if P, then Q; and if Q, then P".

Since both the Pythagorean Theorem and its converse are true, then the associated *if-and-only-if* statement is true. The *if-and-only-if* statement associated to the Pythagorean Theorem and its converse can be stated as follows:

*In a triangle which has sides of length a, b and c:*

$a^2 + b^2 = c^2$ ***if and only if*** *the sides of length a and b form a right angle.*

Another example of a true if-and-only-if statement is:

*An integer n is greater than 4.5* ***if and only if*** *n is greater than 4.*

This statement is true because both the implication

*If an integer n is greater than 4.5, then n is greater than 4*

and its converse

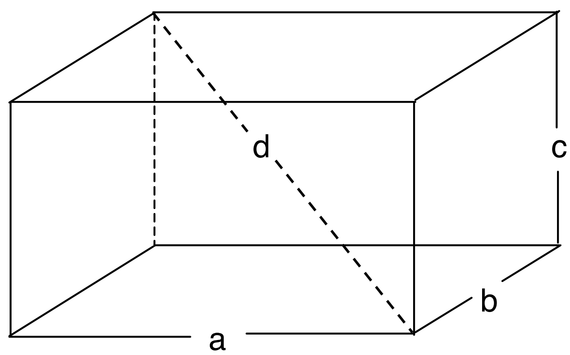
*If an integer n is greater than 4, then n is greater than 4.5*

are true.

### The Pythagorean Theorem in 3 Dimensions.

Suppose a rectangular box has length a, width b and height c. Let d be the length of the diagonal from a corner on the bottom of the box to the opposite corner on the top of the box. Then

$$a^2 + b^2 + c^2 = d^2.$$




**Sample calculation.** My running shoes come in a box that measures 33.5 cm by 20.63 cm by 12.1 cm. Hence, the diagonal of this box is

$$\sqrt{33.5^2 + 20.63^2 + 12.1^2} = \sqrt{1694.2569} = 41.16135202\ldots = 41.2 \text{ cm.}$$

Notice that the two *Calculation Rules* were used in this computation.

**Homework Problem 1. a)** If you construct a triangle out of pieces of spaghetti of lengths 2 cm, 3 cm and 4 cm, is it a right triangle?

**b)** If you want to construct a *right* triangle out of metal rods, and you've already cut two rods of length .6 m and .9 m, how long a rod should you cut for the third side? Is there more than one correct answer? How many correct answers are there?

**c)** Suppose you are going to cut a 10 ft metal rod into 3 pieces (with nothing left over) to make a *right* triangle. Find 5 different ways to do the cuts to make 5 right triangles with different side lengths.

**Homework Problem 2.** Find the Pythagorean triple a, b, c corresponding to the values u = 7, v = 8.

**Homework Problem 3.** Which of the following *if-and-only-if* statements are true?

**a)** A man weighs more than 200 pounds of feathers if and only if he weighs more than 200 pounds of lead.

**b)** The sun doesn't set until 8:30 pm if and only if it is summer.

**c)** 2 times a real number x is 4 if and only if x = 2.

**d)** The square of a real number x is 4 if and only if x = 2.

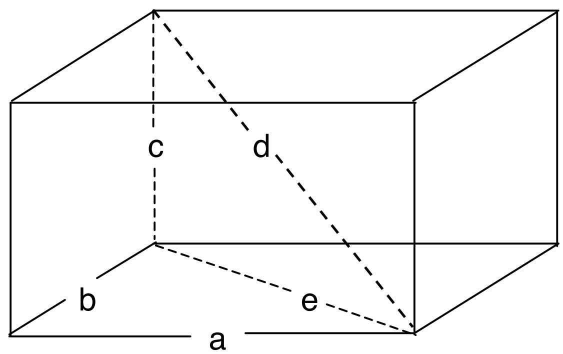
**e)** The area of a square is 9 square feet if and only if the length of each side of the square is 3 feet.

**f)** In a triangle with vertices A, B and C, the angles at A and B both measure 20º if and only if the lengths of the sides joining A to C and B to C are equal.

**Homework Problem 4.** In a large room the length of the diagonal along the floor from the southeast corner to the northwest corner is 24 m. The height of the room's ceiling is 5 m. What is the distance from the southeast corner of the floor to the northwest corner of the ceiling?

**Homework Problem 5.** Try to find a convincing explanation or *proof* of the 3-dimensional Pythagorean Theorem. Write up and hand in a detailed and careful explanation of your proof.

**Hint.** Let e be the length of the diagonal of the bottom of the box and consider the following figure.




**Homework Problem 6.** What is the length of the longest flagpole that can be carried (without bending or breaking) in rectangular container car that has the following interior measurements: 10 ft wide, 8 feet high and 60 ft long.

**Homework Problem 7.** Bart and Lisa work in different office buildings in downtown Chicago. Bart's office is .3 km above the ground, Lisa's office is .5 km above the ground, and Bart's building is 1.1 km south and .7 km east of Lisa's building.

**a)** If a pigeon flies a straight line from Bart's office window to Lisa's office window, how far will it fly?

**b)** If Bart goes by elevator and taxi from his office to Lisa's office, how far will he travel. (The streets of downtown Chicago run either north-south or east-west, and the elevators run up or down.)