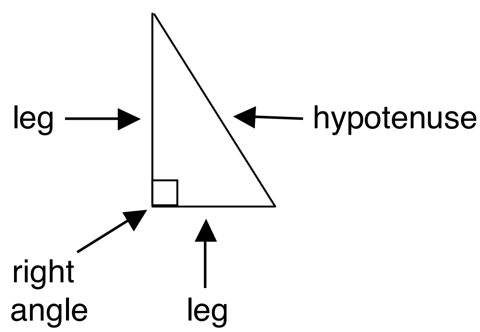
Lesson 2: The Pythagorean Theorem and Similar Triangles

A Brief Review of the Pythagorean Theorem.

Recall that an angle which measures 90º is called a *right angle.* If one of the angles of a triangle is a right angle, then we call the triangle a *right triangle.* In a right triangle, the two sides of the triangle that form the right angle are called the *legs* of the right triangle, and the side of the triangle which is opposite to the right angle is called the *hypotenuse* of the triangle.

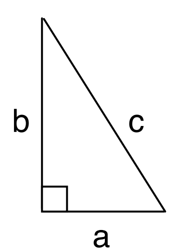


Right Triangle

We are now ready to state:

The Pythagorean Theorem. In a triangle which has sides of lengths a, b and c, if the sides of length a and b form a right angle, then side lengths satisfy the following equation:

$$a^2 + b^2 = c^2.$$

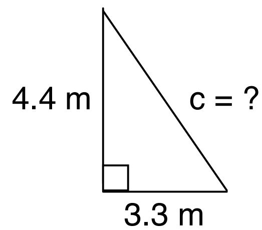


In words, the Pythagorean Theorem says "The square of the hypotenuse is equal to the sum of the squares of the other two sides."

There are many different explanations of why the Pythagorean Theorem is true. These explanations are called *proofs* of the Pythagorean Theorem and we will give them more attention later in this lesson.[1] Here we focus on applications of the Pythagorean Theorem.

We now present three sample applications of the Pythagorean Theorem. In each application, we will use the Pythagorean Theorem to find the unknown side length. When performing these calculations, we will pay careful attention to the units in which the given data is presented, and we will obey the two *Calculation Rules* stated in Lesson 1.

Application 1.



In the triangle shown above, the lengths of two sides are given to the nearest tenth of a meter. In this situation Calculation Rule 2 tells us that our final answer for side length c should be expressed to the nearest tenth of a meter. According to the Pythagorean Theorem:

$$c^2 = (3.3)^2 + (4.4)^2.$$

Hence,

$$c = \sqrt{(3.3)^2 + (4.4)^2}.$$

Using a pocket calculator, we find that

$$\sqrt{(3.3)^2 + (4.4)^2} = 5.5 \text{ meters}.$$

In this calculation, we are fortunate. The answer produced by the calculator requires no rounding. Hence, the solution to this problem is:

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[1] Here are four websites that present a variety of proofs of the Pythagorean Theorem:

http://www.usna.edu/MathDept/mdm/pyth.html

http://www.cut-the-knot.org/pythagoras/index.shtml

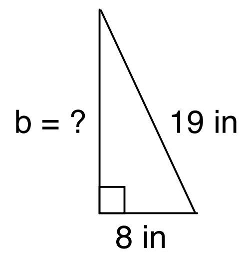
http://www.cut-the-knot.org/pythagoras/PythLattice.shtml

http://www.maa.org/mathland/mathtrek_11_27_00.html

The first site displays an easily understood animated proof. The second site contains 43 different proofs. The third site views the proof in the first site from a more sophisticated perspective that reveals the proof to be one of a family of similar proofs. The fourth site gives a brief history of the Theorem, speculation about how it was discovered, and references to other interesting sites.

c = 5.5 m to the nearest tenth of a meter.

Application 2.



In the triangle shown above, the lengths of two sides are given to the nearest inch. Thus, according to Calculation Rule 2, our final answer for side length b should be expressed to the nearest inch. Here, the Pythagorean Theorem implies that

$$8^2 + b^2 = 19^2.$$

Hence,

$$b^2 = 19^2 - 8^2.$$

Therefore,

$$b = \sqrt{19^2 - 8^2}.$$

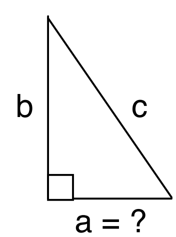
Using a pocket calculator, we find that

$$\sqrt{19^2 - 8^2} = 17.23368794.$$

We round to the nearest inch, to obtain the solution to this problem:

$$b = 17 \text{ in to the nearest inch.}$$

Application 3.



b = 1342 miles

c = 1980 miles to the nearest 10 miles

In the triangle pictured above, the least accurate measurement is in units of 10 miles. Therefore, Calculation Rule 2 stipulates that our final answer should be in units of 10 miles (even though one of the lengths is given in 1 mile units). The Pythagorean Theorem implies that

$$a^2 + 1342^2 = 1980^2.$$

Notice that we do not round the number 1342 to 1340 even though our final answer must be in units of 10 miles. Here we are obeying Calculation Rule 1 which tells us that to preserve accuracy we should not do any rounding until the end of the calculation. Hence,

$$a = \sqrt{1980^2 - 1342^2}.$$

Using a pocket calculator, we find that

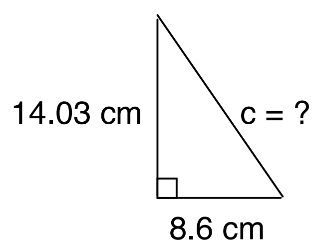
$$\sqrt{1980^2 - 1342^2} = 1455.828287.$$

We round to the nearest 10 miles to obtain the solution to this problem:

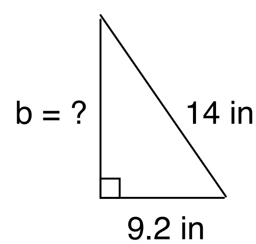
$$a = 1460 \text{ miles to the nearest 10 miles.}$$

Activity 1. Different groups should work on problems **a)** and **b)** below and report their solutions to the class. The class should discuss any discrepancies in the solutions. In each problem, find the unknown side length. Be sure to obey the two *Calculation Rules*.

a)

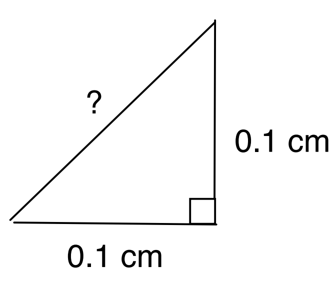


b)



Next we present an illustration of how the *Calculation Rules* save us from error. Suppose we are given a right triangle in which both legs measure .1 cm to the nearest tenth of a centimeter. Let us calculate the length of the hypotenuse of this right triangle. Calculation Rule 2 instructs us to compute the length of the hypotenuse in units of tenths of a centimeter. According to the Pythagorean Theorem the length of the hypotenuse is

$$\sqrt{(.1)^2 + (.1)^2} = \sqrt{.01 + .01} = \sqrt{.02} = .1414\ldots$$



Hence, rounded to the nearest tenth of a centimeter, the length of the hypotenuse of this right triangle is .1 cm. During this computation, we obeyed both Calculations Rules 1 and 2. We didn't round during the calculation, but we rounded at the end of the calculation to make our answer have the same degree of accuracy as the data given at the beginning of the problem.

Suppose we had disobeyed Calculation Rule 1 by rounding the quantity .02 to the nearest tenth of a centimeter before we completed the calculation by taking a square root. This would change .02 to .0. Then the calculation would become:

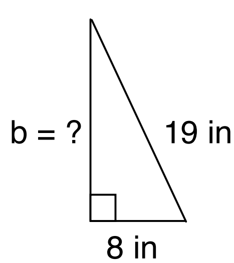
$$\sqrt{(.1)^2 + (.1)^2} \;=\; \sqrt{.01 + .01} \;=\; \sqrt{.02} \;\rightarrow\; \sqrt{.0} \;=\; 0.$$

We would thus arrive at the conclusion that the length of the hypotenuse of this right triangle is 0 cm to the nearest tenth of a centimeter. Since the legs of this triangle are .1 cm each and since the hypotenuse of a right triangle is always longer than its legs, this is an absurd conclusion.

Moral: Don't break the two *Calculation Rules*.

Notice that we have not used the concept of significant figures in the preceding applications of the Pythagorean Theorem. In Lesson 1 we had advised the use of significant figures in calculations involving multiplication or division particularly when input data is expressed in different units. Pythagorean Theorem calculations involve squaring which is a form of multiplication. So it is not unreasonable to think that it would be appropriate to use significant figures in these calculations. However, the method of significant figures is only a rule of thumb to avoid inaccuracy which doesn't always produce optimally accurate results. It turns out that Pythagorean Theorem applications are calculations in which the method of significant figures sometimes creates inaccuracy. So we advise not using significant figures in these kinds of calculations.

We illustrate how the use of significant figures can introduce inaccuracy by reexamining Application 2 above. In application 2 we consider a right triangle in which the hypotenuse has length 19 inches and one leg has length 8 inches, both to the nearest inch. We want to estimate the length b of the other leg. Calculation yields b =



17.23368794, a figure which must be rounded appropriately. Because the input data for this problem was given to the nearest inch, we rounded b to the nearest inch to get b =

17 inches. However, if we were to use the method of significant figures, then because one of the input data – 8 inches – has only one significant figure, we would consequently round b to one significant figure, obtaining b = 20 inches. This estimate is obviously incorrect because the length b of a leg must always be less than the length of the hypotenuse which is 19 inches. To get a more precise idea about the inaccuracy of the estimate b = 20 inches, we perform an interval analysis on this problem. Let a represent the length of the leg which is 8 inches to the nearest inch, and let c represent the length of the hypotenuse – 19 inches to the nearest inch. Then a and c actually lie in the following ranges:

$$7.5 \leq a \leq 8.5 \quad \text{and} \quad 18.5 \leq c \leq 19.5.$$

Since $b = \sqrt{c^2 - a^2}$, then the smallest possible value of b is

$$b = \sqrt{18.5^2 - 8.5^2} = 16.43167673$$

and the largest possible value of b is

$$b = \sqrt{19.5^2 - 7.5^2} = 18.000\ldots$$

Thus, b lies in the range

$$16.43167673 \leq b \leq 18.000.$$

This range comfortably contains our original estimate, b = 17. However, the estimate b = 20 obtain by using significant figures lies well outside this range. The length of this range is 18 – 16.43167673 = 1.56832327. Hence, the estimate b = 20 lies farther above the upper end, 18, of the range than the entire length of the range! Thus, the use of significant figures in this calculation introduces a significant amount of inaccuracy.

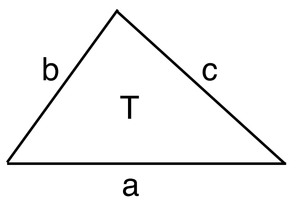
Moral: Don’t use significant figures in Pythagorean Theorem calculations.

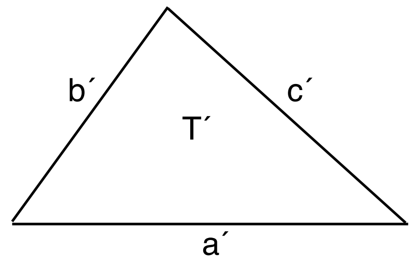
A Brief Review of Similar Triangles

Two triangles T and T´ are *similar* if there is a correspondence between the sides of T and the sides of T´ and there is a real number r > 0 called a *scale factor* such that the length of each side of T´ is equal to r times the length of the corresponding side of T.

In the figure on the next page, suppose that triangle T with side lengths a, b and c is similar to triangle T´ with side lengths a´, b´ and c´ so that the side of T of length a corresponds to the side of T´ of length a´, the side of T of length b corresponds to the side of T´ of length b´ and the side of T of length c corresponds to the side of T´ of length c´. Then there is a scale factor r > 0 such that

$$a' = ra, \quad b' = rb \quad \text{and} \quad c' = rc.$$





We observe that in this situation, the similarity of triangles T and T´ can be expressed without explicitly mentioning the scale factor r. This is because the ratios

$$\frac{a'}{a}, \quad \frac{b'}{b} \quad \text{and} \quad \frac{c'}{c}$$

are all equal to the scale factor r. Hence the three equations

$$a' = ra, \quad b' = rb \quad \text{and} \quad c' = rc$$

can be expressed without mentioning the scale factor r by writing the equations

$$\frac{a'}{a} = \frac{b'}{b} = \frac{c'}{c}.$$

Furthermore, by "cross-multiplying" these equations can be transformed into the equations

$$\frac{a}{b} = \frac{a'}{b'}, \quad \frac{a}{c} = \frac{a'}{c'} \quad \text{and} \quad \frac{b}{c} = \frac{b'}{c'}.$$

Thus, the similarity of triangles T and T´ can be expressed by any one of the following three different *families of similarity equations*:

1) $a' = ra, \quad b' = rb \quad \text{and} \quad c' = rc.$

2) $\frac{a'}{a} = \frac{b'}{b} = \frac{c'}{c}.$

3) $\frac{a}{b} = \frac{a'}{b'}, \quad \frac{a}{c} = \frac{a'}{c'} \quad \text{and} \quad \frac{b}{c} = \frac{b'}{c'}.$

In two of these families of similarity equations, the scale factor doesn't appear. The important lesson for a student confronted with a problem involving similar triangles is: the student should use the family of similarity equations that provides the most help with the solving the problem at hand. The knack of choosing the most helpful family of similarity equations to solve a particular similar triangles problem is a part of the "art" of problem solving.

We make a second observation about this situation. The lengths a, b and c of the sides of triangle T must all be expressed in the *same units*. Similarly, the lengths a´, b´ and c´ of the sides of triangle T´ must all be expressed in the *same units.* However, a, b and c can be expressed in *different units* from a´, b´ and c´! The three families of similarity equations will validly express the similarity of triangles T and T´ even if the side lengths of T are measured in different units than the side lengths of T´. There is no need to convert the side lengths of T and T´ to common units of measure to use the similarity equations. We will solve a problem below which illustrates this observation; in this problem the side lengths of T are expressed in inches, the side lengths of T´ are expressed in centimeters, and the problem is solved without ever converting to a common unit of measure.

According to our definition, the *similarity* of two triangles depends on the ratios of side lengths of the two triangles. There is a surprising and useful geometric fact (or *theorem*) that which allows us to detect the similarity of two triangles, *not* from the lengths of their sides, but instead from the measures of their angles. This theorem, which we call the *Angle Theorem for Similar Triangles* says that if two angles in triangle T have the same *measure* (i.e. size) as two angles in triangle T´, then triangles T and T´ are similar. We state this theorem more precisely:

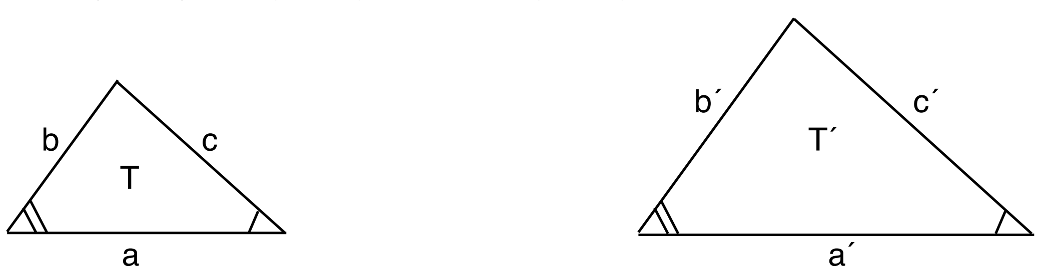
The Angle Theorem for Similar Triangles. Suppose T is a triangle with side lengths a, b and c and T´ is a triangle with side lengths a´, b´ and c´. If the angle in triangle T that is opposite the side of length b has the same measure as the angle in triangle T´ that is opposite the side of length b´, and the angle in triangle T that is opposite the side of length c has the same measure as the angle in triangle T´ that is opposite the side of length c´, then triangles T and T´ are similar, and, in particular, there is a scale factor r > 0 such that

$$a' = ra, \quad b' = rb \quad \text{and} \quad c' = rc.$$

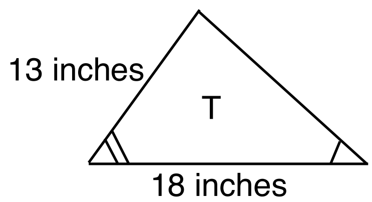
(Then, of course, the other two families of similarity equations also hold:

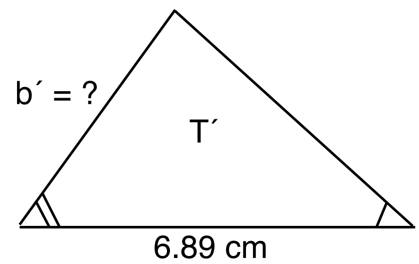
$$\frac{a'}{a} = \frac{b'}{b} = \frac{c'}{c} \text{ and}$$

$$\frac{a}{b} = \frac{a'}{b'}, \quad \frac{a}{c} = \frac{a'}{c'} \quad \text{and} \quad \frac{b}{c} = \frac{b'}{c'}.)$$



We will now show how apply this theorem together with the two *Calculation Rules* to solve a problem. In the figure on the next page, suppose two angles in the triangle T on the left have the same measures as two angles in the triangle T´ on the right (as indicated). The lengths of two sides of T and one side of T´ are given. Find the side length labeled b´ in triangle T´ in centimeters.





Because two angles in T have the same measures as two angles in T´, the Angle Theorem for Similar Triangles implies that T and T´ are similar in such a way that the following equation holds.

$$\frac{b'}{6.89} = \frac{13}{18}.$$

Solving this equation for b´, we get

$$b' = 6.89 \times \frac{13}{18}$$

Using a calculator, we evaluate the expression on the right side of this equation. Following Calculation Rule 1, we don’t round at any point during this calculation. For example, we *don’t* calculate the fraction 13/18 and round it before multiplying by 6.89. This calculation gives us:

$$b' = 4.9761111\ldots$$

Having completed this calculation, we round the final answer following Calculation Rule 2. This calculation is an appropriate place to use significant figures, because the input data is expressed in different units (inches and centimeters) and the calculation involves division. Our value for b´ can be no more accurate than the least accurate input. Since the two inputs – 13 inches and 18 inches – have only two significant figures, then we should round our value for b´ to have only two significant figures. This gives us:

b´ = 5.0 centimeters (to the nearest tenth of a centimeter).

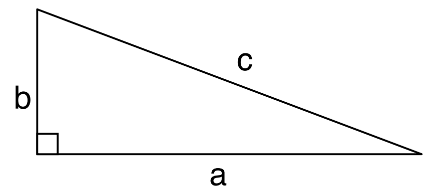
Observe that the side lengths of T are expressed in inches, the side lengths of T´ are expressed in centimeters, and the problem is solved without converting the lengths to a common unit of measure.

Activity 2. Each group should work on the problem stated below and report its solution to the class. The class should discuss any discrepancies in the solutions. Be sure to obey the two *Calculation Rules*.

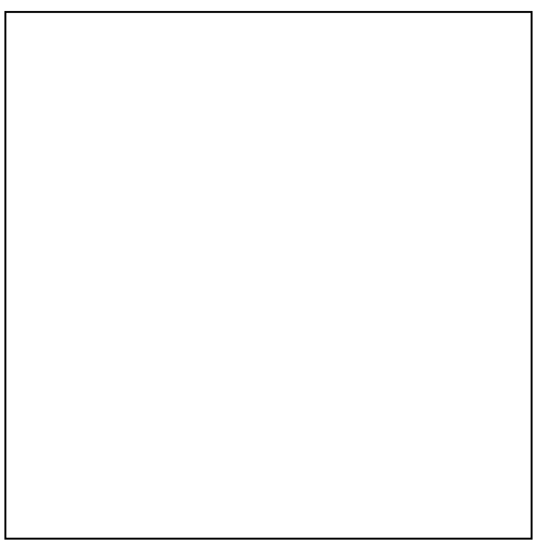
As indicated in the figure below, two angles in the triangle T on the left have the same measures as two angles in the triangle T´ on the right. The lengths of one side of T and two sides of T´ are given. Find the side length labeled b in triangle T in inches.

A Proof of the Pythagorean Theorem

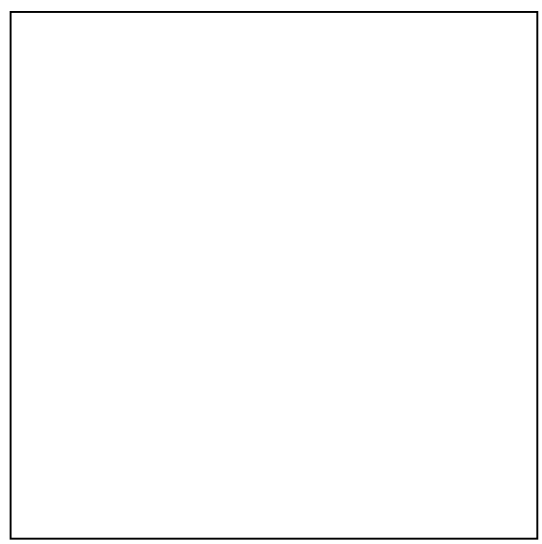
Activity 3. Groups should work on the following problem. This activity is designed to lead you to a proof of the Pythagorean Theorem. Suppose you are given the triangle shown here which has sides of length a, b and c such that the sides of length a and b form a right angle. The following activity is designed to lead you to a proof of the equation $a^2 + b^2 = c^2$.



Below are two squares of the same area, each with sides of length a + b. We call the square on the left “Big Square 1” and the square on the right “Big Square 2”.



Big Square 1

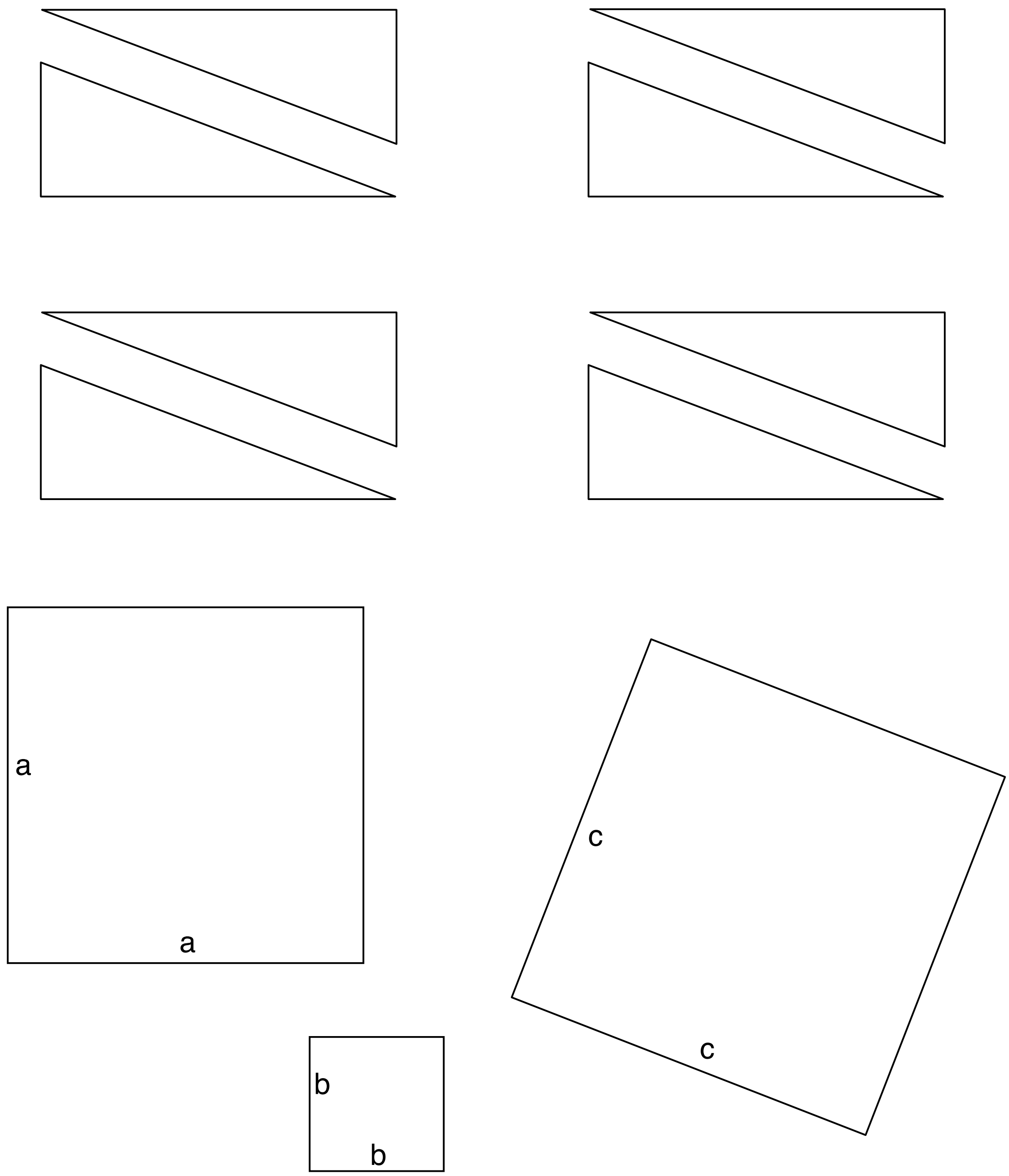


Big Square 2

On the next page are 11 figures: 8 copies of the above triangle together with 3 squares, one of which has sides of length a, another of which has sides of length b, and the third of which has sides of length c. Photocopy this page onto stiffer paper if possible.

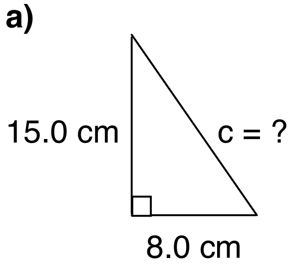
Cut out the 11 figures in the photocopy and arrange them so that they exactly cover the Big Squares 1 and 2 without overlapping. Furthermore, cover Big Square 1 with 4 triangles and the two squares of side lengths a and b, and cover Big Square 2 with 4 triangles and the square of side length c.

Compare the sum of the areas of the figures covering Big Square 1 to the sum of the areas of the figures covering Big Square 2. From this comparison, prove the equation $a^2 + b^2 = c^2$.

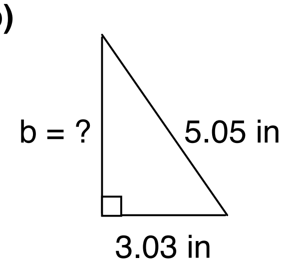


Homework Problem 1. This problem has parts a) through k). In parts a) through h), find the unknown side lengths. Be sure to obey the two *Calculation Rules* in all parts of the problem.

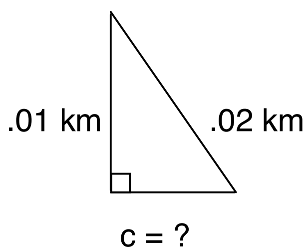
a)



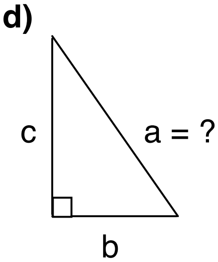
b)



c)



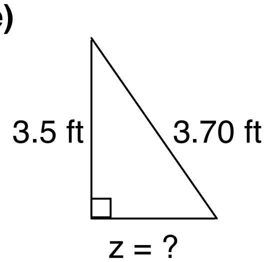
d)



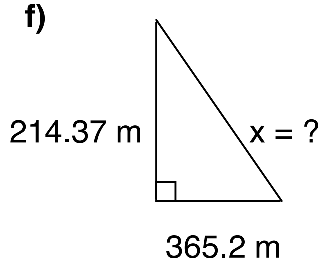
b = 25 m

c = 60 m to the nearest 10m

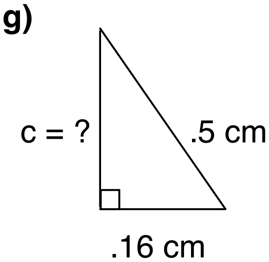
e)



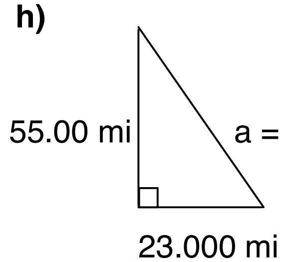
f)



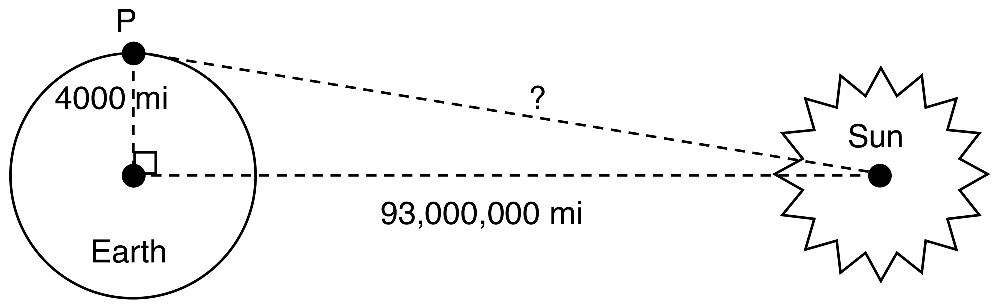
g)



h)



i) The distance from the center of the Earth to the center of the Sun is 93,000,000 miles to the nearest 100,000 miles. The radius of the Earth is 4000 miles to the nearest 100 miles. P is a point on the surface of the Earth with the property that the line joining the center of the Earth to P is perpendicular to the line joining the centers of the Earth and the Sun. How far is P from the center of the Sun?

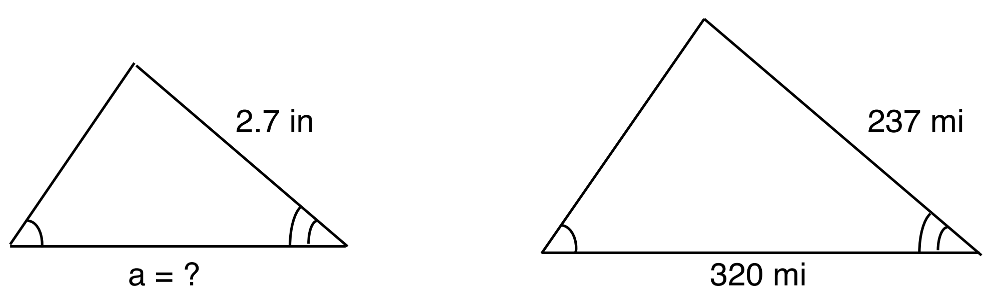


j) A ladder leans against a wall. The foot of the ladder is 3 feet $5^1/_4$ inches from the bottom of the wall to the nearest quarter of an inch. The ladder is 12 feet long to the nearest inch. How high above the ground is the top of the ladder?

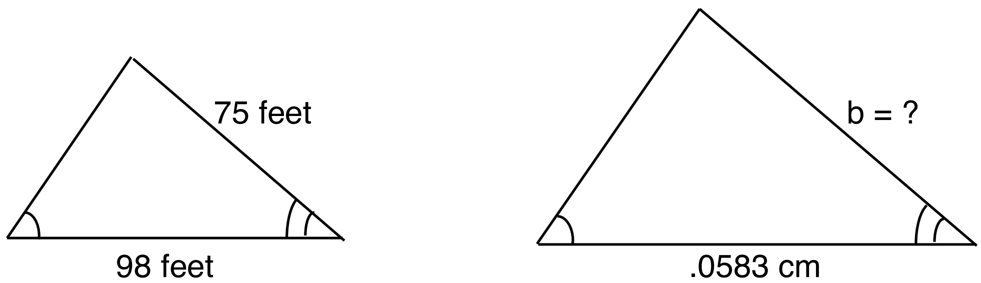
k) On a piece of graph paper draw the x and y axes and label units on these axes so that you can plot the points A = (17,2), B = (2,19) and C = (3,11) on the piece of paper. Calculate the perimeter of the triangle with vertices A, B and C to the nearest tenth of a unit.

Homework Problem 2. In parts a) and b) of this problem, two angles in the triangle on the left have the same measures as two angles in the triangle on the right (as indicated). Find the unknown side lengths. Be sure to obey the two Calculation Rules.

a)



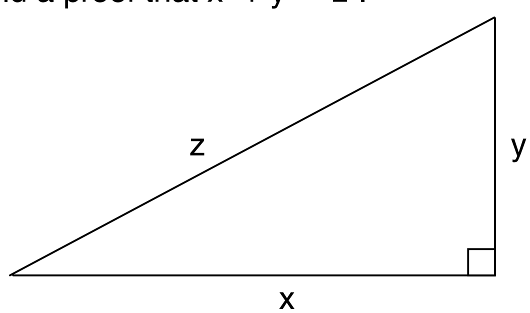
b)



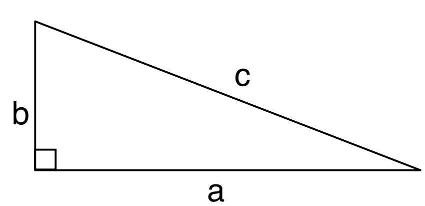
Homework Problem 3.

a) Refer to Activity 3. Write out and hand in a detailed and careful explanation (i.e., a proof) of why $a^2 + b^2 = c^2$. You may use facts about areas of squares and triangles in your proof.

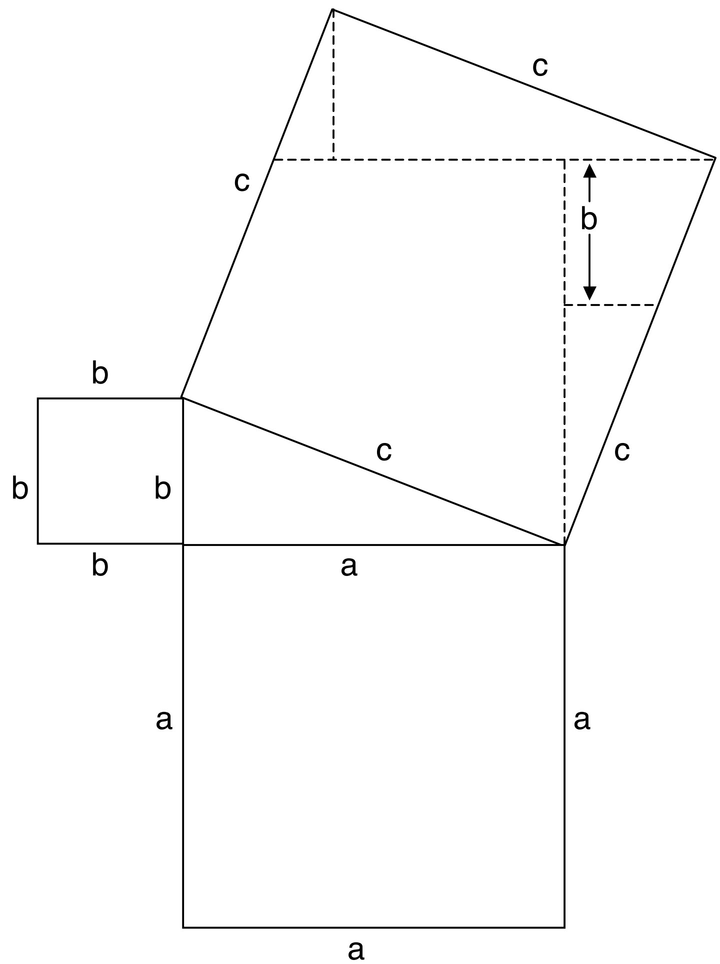
b) Adapt this argument to the triangle with sides of length x, y and z shown below. In other words, draw two squares of side x + y and subdivide them into squares and triangles so that the proof described above for the triangle with side lengths a, b and c can be modified to yield a proof that $x^2 + y^2 = z^2$.



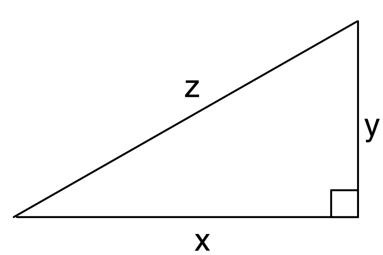
Homework Problem 4. The goal of this problem is also to lead you to a proof of the Pythagorean Theorem. This proof is different from the one presented in Activity 3. Again suppose you are given the triangle shown here which has sides of length a, b and c such that the sides of length a and b form a right angle.



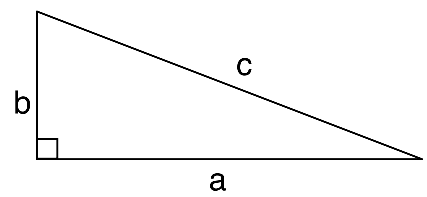
a) In the figure below, we erect three squares on the sides of this triangle. One square has side length a, the second has side length b, and the third has side length c. The square of side length c is subdivided by dotted lines. Photocopy this diagram onto stiffer paper if possible. Cut out the square of side length c and cut it apart along the dotted lines to obtain five pieces. Show that the five pieces can be arranged to cover the two squares of side lengths a and b with no overlap. Argue that this proves $a^2 + b^2 = c^2$.



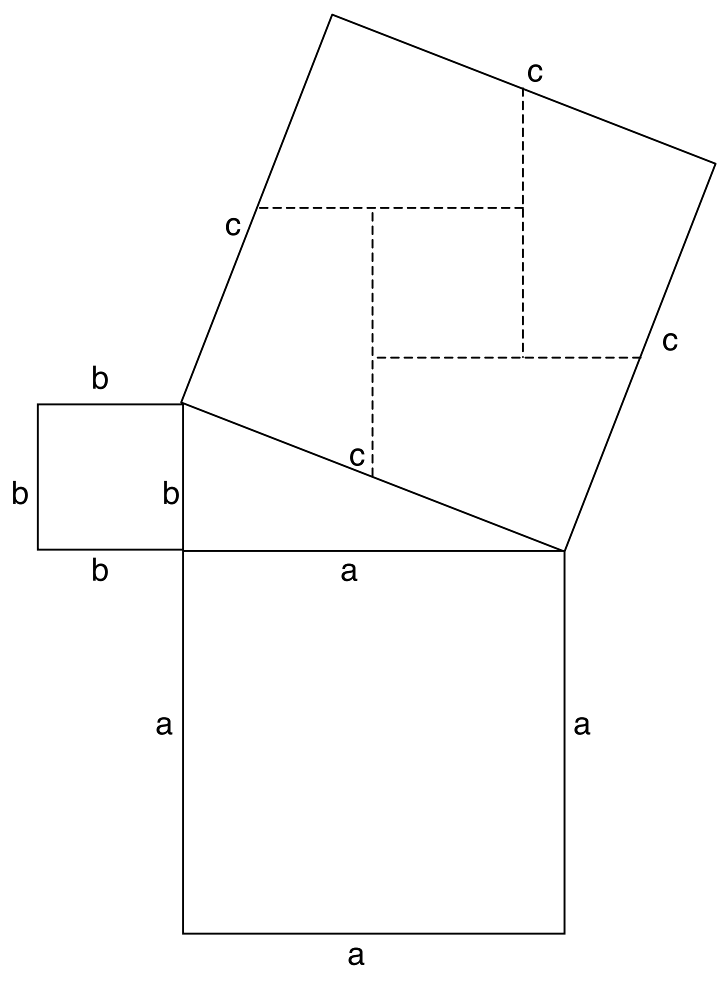
b) Adapt this argument to the triangle with sides of length x, y and z shown below. In other words, erect 3 squares on the sides of this triangle and show how to subdivide the square of side length z into five pieces that will exactly cover the two squares of side lengths x and y.



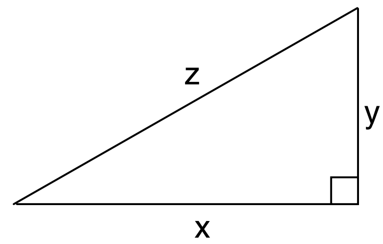
Homework Problem 5. This problem leads to yet another proof of the Pythagorean Theorem. This proof is similar to the previous one. Again start with the triangle shown here with side lengths a, b and c and a right angle between the sides of length a and b.



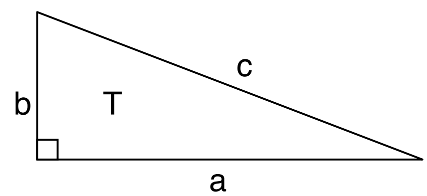
a) Again, in the figure below, we erect three squares on the sides of this triangle. This time the big square with side length c is subdivided by four dotted line segments. These dotted line segments are determined by the rule that they are either horizontal or vertical and one endpoint of each dotted line segment is the *midpoint* of a side of the big square. Again, photocopy this diagram onto stiffer paper if possible, and cut the big square apart along the dotted line segments to obtain five pieces, and show that the five pieces can be arranged to cover the two smaller squares with no overlap. Argue that this proves $a^2 + b^2 = c^2$.



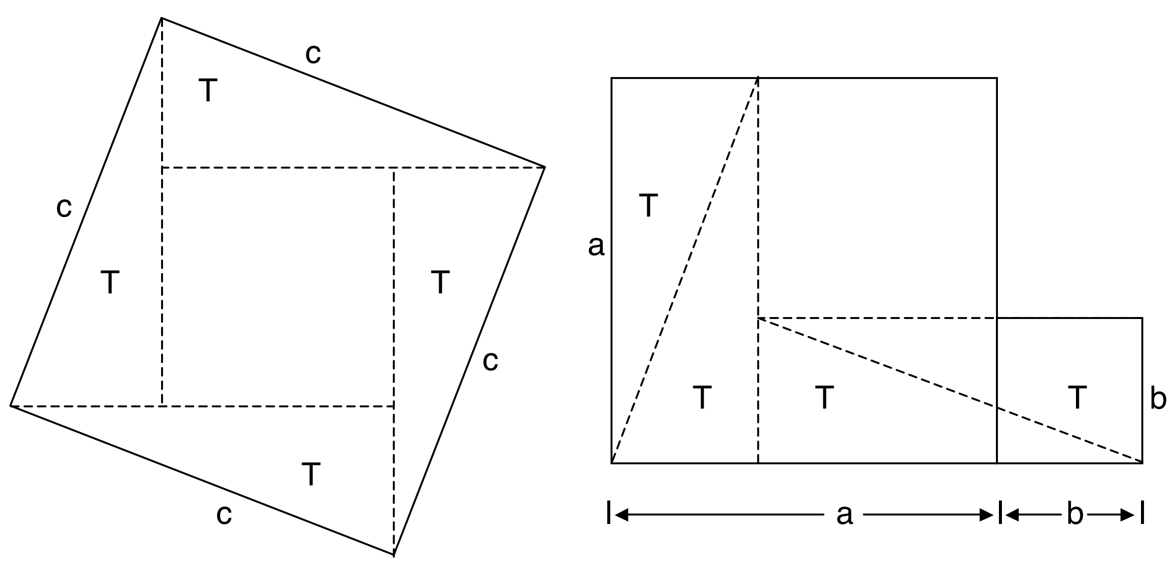
b) Adapt this argument to the triangle with sides of length x, y and z shown below.



Homework Problem 6. The goal of this problem is to lead you to a proof of the Pythagorean Theorem that was known to the ancient Chinese about 1000 BC. (Pythagoras lived about 500 BC.) Suppose you are given the triangle T shown here which has sides of length a, b and c such that the sides of length a and b form a right angle.



a) Consider the following two figures. The figure on the left is a square which has sides of length c. The figure on the right is the union of two squares, one of which has sides of length a and the other of which has sides of length b. Each figure is subdivided by dotted lines into 5 pieces: 4 copies of the original triangle T and a square. Compare the sum of the areas of the pieces making up the square on the left one with the sum of the areas of the pieces making up the figure on the right. Try to derive the equation $a^2 + b^2 = c^2$. (Hint: Each figure contains 4 triangles and a square. How long are the sides of these squares?)



b) Adapt this argument to the triangle with sides of length x, y and z shown below.