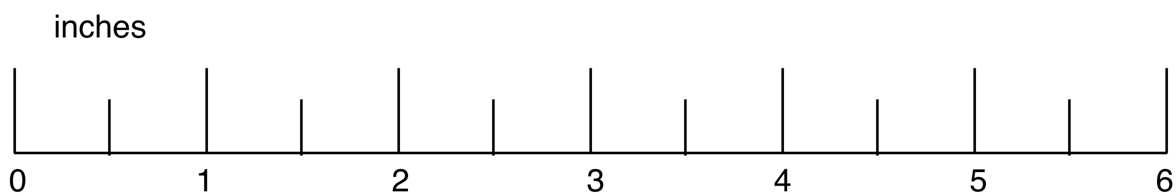
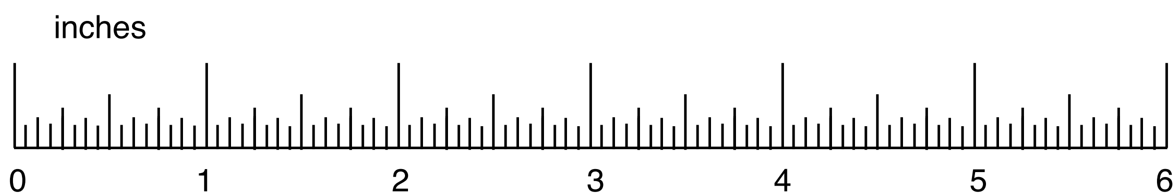
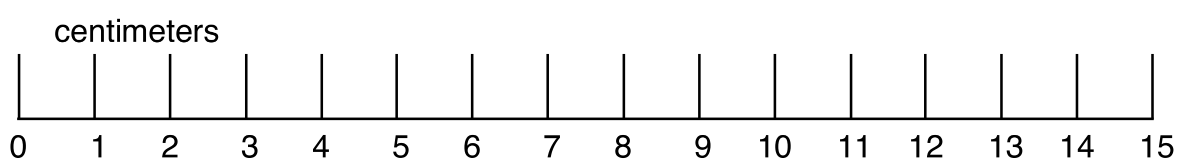
Lesson 1: Measurement and Accuracy

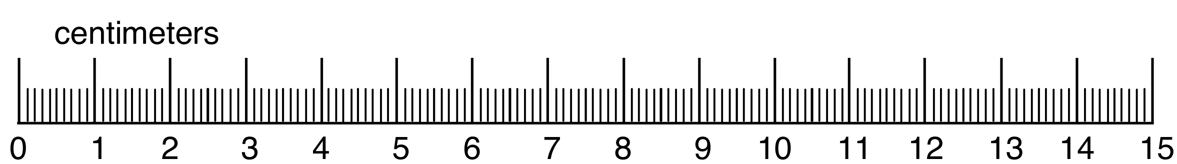
Activity 1. a) Each group will use the four rulers shown below to measure and record the length of the line segment drawn at the bottom of the page. (Copy the line segment onto a piece of patty paper to perform the measurement.) When recording the measurement made with a particular ruler, each group should decide what degree of accuracy (i.e., smallest unit of measure) is appropriate for the recorded measurement with the given ruler.

b) Individuals and groups will report their measurements to the class, and the class will discuss and try to account for discrepancies between these measurements.









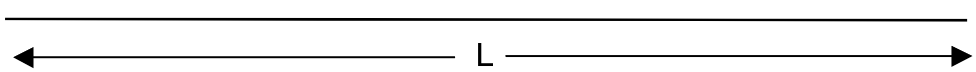
Remarks on units of measure, accuracy of measurements, and rounding. We will use either *metric* units (meter, centimeter, kilometer, etc.) or *English* or *customary* units (foot, inch, mile, etc.) to measure length.[1] We will use the standard abbreviations for these units:

meter = m, centimeter = cm, kilometer = km, foot = ft, inch = in, mile = mi

Here are the basic facts about converting between metric and English units.[2]

1 in = 2.54 cm	1 cm = .3937 in
1 ft (= 12 in) = .3048 m	1 m (= 100 cm) = 3.28 ft = 39.37 in
1 mi (= 5280 ft) = 1.609 km	1 km (= 1000 m) = .621 m

What does it means to measure the length of a line segment to a certain degree of accuracy? Suppose that we measure the length L of the line segment shown below with a ruler that is graduated in units of tenths of a centimeter (= millimeters), and we find that the length L of this segment is closer to the 12.7 centimeter mark than it is to



any other mark on the ruler. Then we can express this fact by writing

L = 12.7 cm *to the nearest tenth of a centimeter".*

However, people often don't bother to include the expression "to the nearest tenth of a centimeter". Instead they simply write

"L = 12.7 cm".

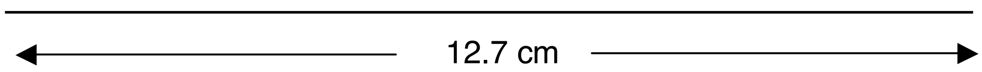
So it is up to us to interpret the expression "L = 12.7 cm" to mean that the length L of the segment is 12.7 cm to the nearest tenth of a centimeter. In other words, "L = 12.7 cm" means that the length L of the segment is closer to the 12.7 cm mark on the ruler than it is to either the 12.6 cm mark or the 12.8 cm mark.

The point of the previous paragraph is that if we encounter either a statement which says that the length of a certain line segment is 12.7 cm or a labeled picture like

---

[1] An excellent website for information about many different units of measure is *http://www.unc.edu/~rowlett/units/*. Within this website, the webpage *http://www.unc.edu/~rowlett/units/custom.html* describes the origin and history of the English or customary units of measure.

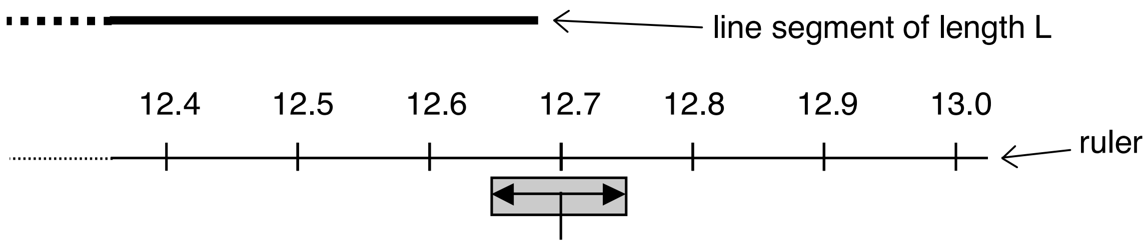
[2] The website *http://www.google.com* can be used as a calculator to do unit conversion. For example, to find the number of kilometers that is equivalent to 35 miles, type "35 mi in km" in the Google search window and click the search button. The website will respond with the answer: 35 mi = 56.32704 kilometers. (You can also type "35 miles in kilometers"; abbreviations aren't necessary.)



then we should interpret this statement or picture to mean that the length L of this segment is closer to the 12.7 cm mark on the ruler than it is to either the 12.6 cm mark or the 12.8 cm mark. Such a statement or picture does not determine the length L of the line segment precisely. It only restricts the value of L to a range. In this case, the range is

$$12.65 \le L \le 12.75,$$

because this is the range of numbers that are closer to 12.7 than they are to 12.6 or 12.8. Sometimes the restriction $12.65 \le L \le 12.75$ is expressed by writing $L = 12.7 \pm .05$ cm.



L = 12.7 cm
to the nearest tenth of a centimeter

Now suppose that we measure the length L of the line segment shown above with a ruler that is more finely graduated into units of hundredths of a centimeter, and we find that the length is closer to the 12.69 centimeter mark than it is to any other mark on the ruler. (We might need a magnifying glass to carry out this measurement.) In this case we would write

"L = 12.69 cm *to the nearest hundredth of a centimeter".*

This means that the length L of the segment is closer to the 12.69 cm mark on the ruler than it is to either the 12.68 cm mark or the 12.70 cm mark. In other words, it means that the length L lies in the range

$$12.685 \le L \le 12.695.$$

Again, this restriction can also be expressed by writing

$$L = 12.69 \pm .005 \text{ cm}.$$

Of course, someone might express the result of this measurement by simply writing

"L = 12.69 cm"

expecting you to supply the additional information that the measurement is "to the nearest hundredth of a centimeter".

Next suppose we measure the length L of this segment with a ruler that is graduated more coarsely into centimeter units, and we find that the length is closer to the 13 centimeter mark than it is to any other mark on the ruler. Then we would write

"L = 13 cm *to the nearest centimeter*".

(Again someone might express this measurement by writing simply

"L = 13 cm"

expecting you to fill in "to the nearest centimeter".) Either of these statements means that the length L of the segment is closer to the 13 cm mark on the ruler than it is to either the 12 cm mark or the 14 cm mark. In other words, they mean that the length L lies in the range

$$12.5 \le L \le 13.5.$$

This can also be expressed by writing

$$L = 13 \pm .5 \text{ cm}.$$

(Notice that according to the interpretation suggested here, the simple statements "L = 13 cm" and "L = 13.0 cm" have different meanings. The statement "L = 13 cm" means that the length L of the segment has been measured using a ruler graduated in centimeters and has been found to be closer to the 13 cm mark than to either the 12 cm mark or the 14 cm mark. In other words, the statement "L = 13 cm" restricts L to the range $12.5 \le L \le 13.5$. On the other hand, the statement "L = 13.0 cm" means that the length L of the segment has been measured using a ruler graduated in tenths of a centimeter and has been found to be closer to the 13.0 cm mark than to either the 12.9 cm mark or the 13.1 cm mark. In other words, the statement "L = 13.0 cm" restricts L to the range $12.95 \le L \le 13.05$.)

Finally suppose that we measure the length L of a different line segment with a very long but crudely marked ruler that is graduated into 10 centimeter units, and we find that the length is closer the 140 centimeter mark than to any other mark on the ruler. Then we would write

"L = 140 cm to the nearest 10 centimeters".

In this case we would ***not*** write "L = 140 cm" because this expression would usually be interpreted to mean L = 140 cm to the nearest centimeter, not the nearest 10 centimeters. In this case, we do not have the luxury of omitting the expression "to the nearest 10 centimeters". The statement "L = 140 cm to the nearest 10 centimeters" means that L is closer to the 140 cm mark on the ruler than it is to either the 130 cm mark or the 150 cm mark. So this statement restricts L to the range

$$135 \le L \le 145,$$

which can also be written

$$L = 140 \pm 5 \text{ cm.}$$

(On the other hand, the statement "L = 140 cm" is usually interpreted to mean that L is closer to 140 cm than it is to 139 cm or 141 cm. So "L = 140 cm" restricts L to the range $139.5 \le L \le 140.5$.)

Unfortunately, authors sometimes fail to specify the degree of accuracy of measurements. A common example is the statement "the circumference of the Earth is 40,000 km". This might mean

a) "the circumference of the Earth is 40,000 km to the nearest 10,000 km",

or it might mean

b) "the circumference of the Earth is 40,000 km to the nearest 1000 km",

or it might mean

c) "the circumference of the Earth is 40,000 km to the nearest 100 km",

⋮

In other words, the statement "the circumference of the Earth is 40,000 km" might mean

a) "the circumference of the Earth is closer to 40,000 km than it is to 30,000 km or 50,000 km",

or it might mean

b) "the circumference of the Earth is closer to 40,000 km than it is to 39,000 km or 41,000 km",

or it might mean

c) "the circumference of the Earth is closer to 40,000 km than it is to 39,900 km or 40,100 km",

⋮

The original statement "the circumference of the Earth is 40,000 km" doesn't clarify which of the three statements a), b) or c) is true. In fact, the circumference of the Earth (at the equator) is 40,075 km to the nearest kilometer.[3] So statements a) and b) are

---

[3] The Earth is not a perfect sphere. It bulges at the equator and is slightly flattened at the poles. As a result, its equatorial circumference is slightly larger than its polar circumference: 40,075 km around the equator versus 40,008 km through the poles to the nearest km. The precise value of the Earth's

true, but statement c) is false.

Activity 2. Different groups should work on problems **a)** through **f)** on the next three pages and report their solutions to the class. The class should discuss any discrepancies in the solutions.

Suppose that a person measures the length L of a line segment with a ruler and then fills in the blanks in one of the statements **1)** through **4)** or in diagram **5)** below. Given this information, your job is to fill in all the other blanks consistently. Each of problems **a)** through **f)** on the next three pages poses an exercise of this type.

1) The ruler is graduated in units of ______________________________

and we find that L is closer to the _____________ mark on the ruler than it is to

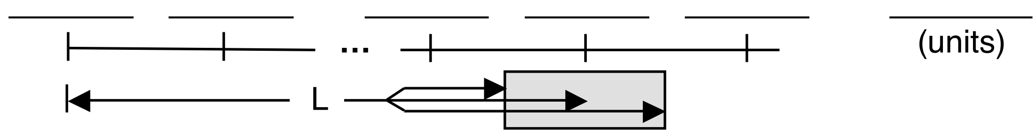
either the _____________ mark or the _____________ mark.

2) L = ________________ to the nearest ________________________.

3) ________________ ≤ L ≤ ________________.

4) L = ________________ ± ________________________.

5)



---

circumference and diameter is of great interest to scientists. (Recall that the circumference of a circle is equal to π times its diameter.) For example, scientists who study climate change calculate the rise in sea level using data from satellites. The accuracy of their calculations depends on knowing the exact positions of the satellites which in turn depends on having a value for the Earth's diameter that is correct to the nearest millimeter. The precision of these estimates is constantly being improved. In July 2007 a new extremely accurate estimate of the Earth's diameter was published that decreased the accepted value by 5 millimeters (.2 inch). This estimate was obtained by a technique called Very Long Baseline Interferometry that relies on a worldwide network of more than 70 radio telescopes that receive radio waves emitted by distant galaxies. Telescopes in different locations receive the same signal at slightly different times, and these timelags facilitate extremely precise estimates of the Earth's diameter.

a) Suppose that statement **4)** reads "L = 19.34 ± .005 cm." Fill in the blanks in statements **1)**, **2)** and **3)** and in diagram **5)** to make them have the same meaning as statement **4).**

1) The ruler is graduated in units of ________________________________

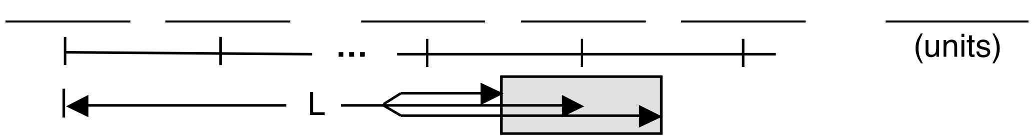
and we find that L is closer to the ______________ mark on the ruler than it is to

either the ______________ mark or the ______________ mark.

2) L = _________________ to the nearest _________________________.

3) _________________ ≤ L ≤ _________________.

5)



b) Suppose statement **2)** reads "L = $8^{1}/_{2}$ inches to the nearest quarter of an inch". Fill in the blanks in statements **1)**, **3)** and **4)** and in diagram **5)** to make then have the same meaning as statement **2)**.

1) The ruler is graduated in units of ________________________________

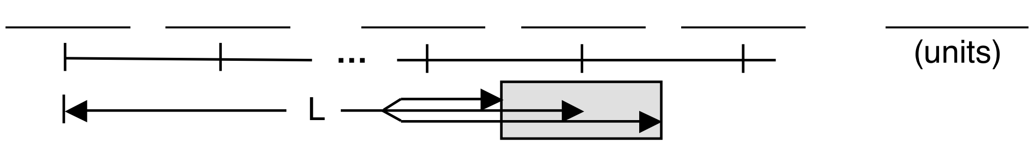
and we find that L is closer to the ______________ mark on the ruler than it is to

either the ______________ mark or the ______________ mark.

3) _________________ ≤ L ≤ _________________.

4) L = _________________ ± _________________________.

5)



c) Suppose statement **2)** reads "L = <u>20,000 millimeters</u> to the nearest <u>100 millimeters</u>". Fill in the blanks in statements **1), 3)** and **4)** and in diagram **5)** to make then have the same meaning as statement **2)**.

1) The ruler is graduated in units of ________________________________

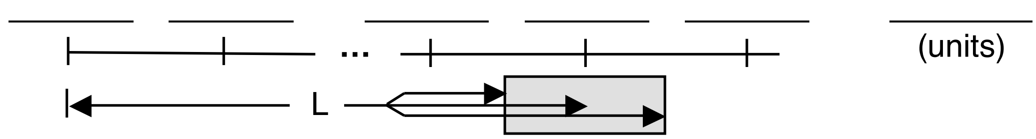
and we find that L is closer to the ______________ mark on the ruler than it is to

either the ______________ mark or the ______________ mark.

3) _________________ ≤ L ≤ _________________.

4) L = _________________ ± _________________________.

5)



d) Suppose statement **3)** reads "<u>48,500 nanometers</u> ≤ L ≤ <u>49,500 nanometers</u>". Fill in the blanks in statements **1), 2)** and **3)** and in diagram **5)** to make them have the same meaning as statement **4)**. (1 *nanometer* = 1 billionth of a meter.)

1) The ruler is graduated in units of ________________________________

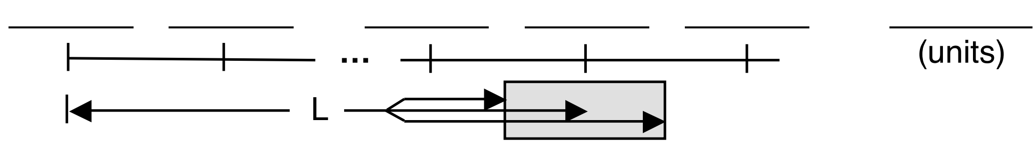
and we find that L is closer to the ______________ mark on the ruler than it is to

either the ______________ mark or the ______________ mark.

2) L = _________________ to the nearest _________________________.

4) L = _________________ ± _________________________.

5)



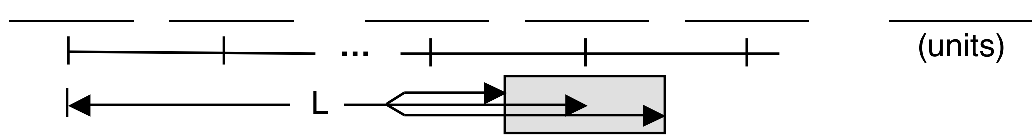
e) Suppose statement **1)** reads "The ruler is graduated in units of <u>sixteenths of an inch</u> and we find that L is closer to the <u>$4^{5}/_{16}$ inch</u> mark than it is to either the <u>$4^{1}/_{4}$ inch</u> mark or the <u>$4^{3}/_{8}$ inch</u> mark." Fill in the blanks in statements **2)**, **3)** and **4)** and in diagram **5)** to make then have the same meaning as statement **1)**.

2) L = ____________________ to the nearest ______________________________.

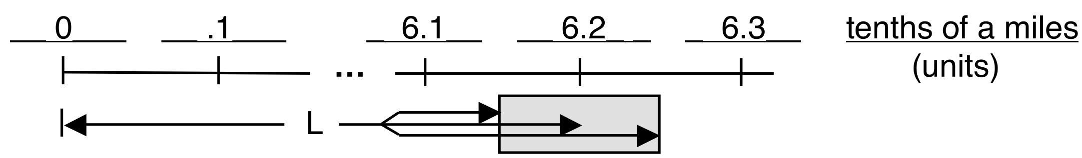
3) ____________________ ≤ L ≤ ____________________.

4) L = ____________________ ± ______________________________.

5)



f) Suppose diagram **5)** reads



Fill in the blanks in statements **1), 2), 3)** and **4)** to make them have the same meaning as diagram **5)**.

1) The ruler is graduated in units of _____________________________________

and we find that L is closer to the ________________ mark on the ruler than it is to either the ________________ mark or the ________________ mark.

2) L = ____________________ to the nearest ______________________________.

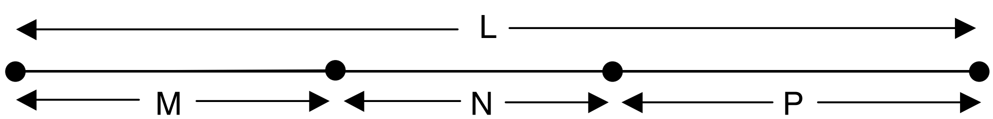
3) ____________________ ≤ L ≤ ____________________.

4) L = ____________________ ± ______________________________.

Notice that the interpretation suggested here for statements like "the length L of a certain line segment is 12.7 cm to the nearest tenth of a centimeter" unavoidably introduces some *uncertainty* or *inaccuracy* or *variability* into the value of L. Indeed, this statement really means that L can take on any value in the range between 12.65 cm and 12.75 cm. This uncertainty becomes significant when we perform calculations based on known length measurements to discover the value of unknown length measurements. The problem is that calculations tend to increase the degree of uncertainty associated to length measurements.

We illustrate the phenomenon that calculations increase uncertainty with a simple example involving addition. Suppose that a line segment of unknown length L is divided into three smaller line segments of known lengths

M = 3.5 cm, N = 2.7 cm and P = 4.1 cm (to the nearest tenth of a centimeter).



Then we can compute

$$L = M + N + P = 3.5 + 2.7 + 4.1 = 10.3 \text{ cm}.$$

How accurately do we know the value of L? To answer this question, observe that the values of M, N, and P can be expressed as

$$M = 3.5 \pm .05 \text{ cm}, \quad N = 2.7 \pm .05 \text{ cm}, \quad \text{and} \quad P = 4.1 \pm .05 \text{ cm}.$$

So the minimum possible values of M, N, and P are

$$M = 3.45 \text{ cm}, \quad N = 2.65 \text{ cm}, \quad \text{and} \quad P = 4.05 \text{ cm};$$

and the maximum possible values of M, N, and P are

$$M = 3.55 \text{ cm}, \quad N = 2.75 \text{ cm}, \quad \text{and} \quad P = 4.15 \text{ cm}$$

Hence, the minimum possible value of L = M + N + P is

$$3.45 + 2.65 + 4.05 = 10.15 \text{ cm},$$

and the maximum possible value of L is

$$3.55 + 2.75 + 4.15 = 10.45 \text{ cm}.$$

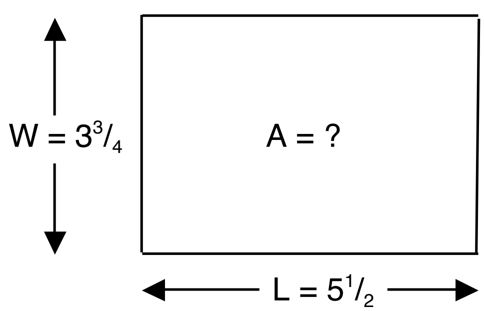
Equivalently, L lies in the range

$$10.15 \leq L \leq 10.45.$$

This inequality tells us that the value of L falls somewhere in a interval of length .3 cm. Hence, all that we can say about the value of L to the nearest tenth of a centimeter is that it is either 10.2, 10.3 or 10.4. This conclusion seems to be at odds with our original calculation which showed that L = 10.3 cm. What has happened here is that the addition operations performed during the calculation have inflated the magnitude of the uncertainty associated to the length measurements. The original length measurements M, N and P have associated ranges of uncertainty of length .1 cm. The length measurement L which is the result of the calculation has an associated range of uncertainty of length .3 cm.

The method we have just illustrated for determining the inaccuracy of the computed length L is often called *interval analysis* or *interval arithmetic* because it tells us that the value of L must lie in the interval between 10.15 cm and 10.45 cm.

We give a second simple application of interval analysis to show of how calculation increases uncertainty. This example involves multiplication. Suppose that a rectangle has length $L = 5^1/_2$ inches to the nearest quarter of an inch and width $W = 3^3/_4$



inches to the nearest quarter of an inch. Then the area A of this rectangle is

$$A = L \times W = 5^1/_2 \times 3^3/_4 = {}^{11}/_2 \times {}^{15}/_4 = {}^{165}/_8 = 20^5/_8 \text{ square inches.}$$

We know the length L and width W of this rectangle to the nearest quarter of an inch. How accurately do we know the value of the area A of this rectangle? Knowing $L = 5^1/_2$ inches and $W = 3^3/_4$ inches to the nearest quarter of an inch means we can write

$$L = 5^1/_2 \pm {}^1/_8 \text{ in} \quad \text{and} \quad W = 3^3/_4 \pm {}^1/_8 \text{ in}$$

So the minimum possible values of L and W are

$$L = 5^3/_8 \text{ in} \quad \text{and} \quad W = 3^5/_8 \text{ in,}$$

and the maximum possible values of L and W are

$$L = 5^5/_8 \text{ in} \quad \text{and} \quad W = 3^7/_8 \text{ in.}$$

Hence the minimum possible value of A = L x W is

$$5^3/_8 \times 3^5/_8 = {}^{43}/_8 \times {}^{29}/_8 = {}^{1247}/_{64} = 19^{31}/_{64} \text{ square inches}$$

and the maximum possible value of A is

$$5\,^5/_8 \times 3\,^7/_8 \;=\; ^{45}/_8 \times {}^{31}/_8 \;=\; ^{1395}/_{64} \;=\; 21\,^{51}/_{64} \text{ square inches.}$$

Equivalently, A lies in the range

$$19\,^{31}/_{64} \;\le\; A \;\le\; 21\,^{51}/_{64}.$$

This inequality tells us that the area A of the rectangle falls somewhere in a interval of length

$$21\,^{51}/_{64} - 19\,^{31}/_{64} \;=\; 2\,^5/_{16} \text{ square inches.}$$

The multiplication operation performed during this calculation has inflated the magnitude of the uncertainty associated to the measurements. The original length measurements L and W have associated ranges of uncertainty of length $^1/_4$ inch. The area measurement A which is the result of the calculation has an associated range of uncertainty of length $2\,^5/_{16}$ square inches.

Our third and final application of interval analysis to show how calculation increases uncertainty involves division. Suppose that a rectangle has length L = 4.7 inches to the nearest tenth of an inch and area A = 32.35 square inches to the nearest hundredth of a square inch. (Since a square that is $^1/_{10}$ inch on each side has area $^1/_{100}$ square inch, then the degrees of accuracy of L and A are appropriately related.) Then the width of W of this rectangle is

$$W \;=\; \frac{A}{L} \;=\; \frac{32.35}{4.7} \;=\; 6.88\ldots \text{ inches.}$$

Again we ask: how accurately do we know the value of the width W? The given values of L and A allow us to write

$$L \;=\; 4.7 \pm .05 \text{ in} \quad \text{and} \quad A \;=\; 32.35 \pm .005 \text{ sq in.}$$

So the minimum possible value of A is 32.345 sq in and the maximum possible value of L is 4.75 inch. Hence, the minimum possible value of W is

$$W \;=\; \frac{A}{L} \;=\; \frac{32.345}{4.75} \;=\; 6.809\ldots \text{ inches.}$$

(**Question:** Why do we use the minimum value of A and the maximum value of L to compute the minimum value of $W = \dfrac{A}{L}$?)

Similarly, the maximum possible value of A is 32.355 sq in and the minimum possible value of L is 4.65 inches. Hence, the maximum possible value of W is

$$W \;=\; \frac{A}{L} \;=\; \frac{32.355}{4.65} \;=\; 6.958\ldots \text{ in.}$$

Therefore, W lies in the range

$$6.809 \leq W \leq 6.958.$$

This inequality implies that the width W of the rectangle falls somewhere in an interval of length

$$6.958 - 6.809 = .149 \text{ inches.}$$

Thus, the division operation performed during this calculation has inflated the magnitude of the uncertainty associated to the measurements. The original length and area measurements L and W have associated ranges of uncertainty of lengths .1 in and .01 sq in, respectively. Our calculation produces a width measurement W that has an associated range of uncertainty of length .149 inches.

The phenomenon that we have just illustrated – the growth of measurement uncertainty during a calculation – is a serious problem. It must be understood and controlled if we are to rely on calculations based on measurements. Fortunately, this problem can be understood and controlled using interval analysis. However, the methods of interval analysis are *not* the subject of this course. We will not explore them in any additional detail. (Further information about interval analysis can be found at *http://en.wikipedia.org/wiki/Interval_arithmetic*.) Instead, we will propose two *Calculation Rules* that should be followed during calculations to minimize the growth of uncertainty. There is no way to stop the growth of uncertainty during calculations, but following these two rules usually prevent *unnecessary* uncertainty from creeping into calculations.

Before we state the two *Calculation Rules*, we must discuss the issue of *rounding*. Suppose that we want to express the result of a measurement or calculation in terms of a certain unit, say tenths of a centimeter. Then we must round the result of the measurement or calculation to the nearest tenth of a centimeter. For example,

12.45118 cm   should be rounded to   12.5 cm

and

12.44997 cm   should be rounded to   12.4 cm.

This is because 12.45118 is closer to 12.5 than to 12.4, while 12.44997 is closer to 12.4 than to 12.5. Indeed, 12.45 is exactly midway between 12.4 and 12.5 and

$$12.44997 < 12.45 < 12.45118.$$

This procedure for rounding to the nearest unit works except in the case of a number that is exactly midway between two units. For example, if our unit is tenths of a centimeter, then the number 12.450…. cm is exactly midway between the two units 12.4 cm and 12.5 cm. In this case, there is no purely mathematical way to decide whether to round down to 12.4 or up to 12.5. In this situation we need a *rounding convention.* A *convention* is a rule which tells us what to do if there is no sound mathematical reason for making a choice. The *convention* establishes common practice in the absence of compelling logic. We need an established convention to tell us not only how we should proceed to carry out a calculation in the event that we must round a number that is midway between two units. We also need an established convention to help us decide, when we are looking at the result of someone else's calculation (say, one of your students), whether they performed the calculation correctly.

The usual rounding convention and the one we will use in this course is: **round up**. In the case that our unit is tenths of a centimeter, then this rounding convention would tell us to round 12.450… cm up to 12.5 cm.

This rounding convention has a disadvantage. Over the long run, it makes our calculations biased to the high side, since any calculation in which this rounding convention is used is half a unit too large.

We could of course choose the alternative rounding convention: **round down**. (This convention would tell us to round 12.450… cm down to 12.4 cm.) However, rounding down has the disadvantage that over the long run, it makes our calculations biased to the low side, since any calculation in which this rounding convention is used is half a unit too small. There is no inherent mathematical reason to choose rounding up over rounding down. But since rounding up is the common, established and expected practice, we stick to it to avoid confusion.

There is an alternative rounding procedure that is sometimes used which differs from the conventions of always rounding up or always rounding down and which mitigates some of the bias inherent in these two conventions. We will not use this alternative, but we describe it to make students aware of its existence. In this alternative procedure, when presented with a number that is midway between two units, we **round up** whenever the digit to be rounded is **odd**, and we **round down** whenever the digit to be rounded is **even.** Thus, when rounding the nearest tenth of a centimeter, we would round 12.450… cm to 12.4 cm (because 4 is even), and we would round 12.750… cm to 12.8 cm (because 7 is odd).

To repeat: we will not use the procedure described in the previous paragraph. In this class, we will follow the usual rounding convention. When presented with a number that must be rounded, we will round the number to the closest unit if it is closer to one unit than it is to any other. However, if the number is midway between two units, we will follow the convention of **always rounding up**.

Having discussed rounding conventions, we are now ready to state the two *Calculation Rules* that will help to keep unnecessary uncertainty from entering our calculations.

Calculation Rule 1: In performing a calculation, *don't round the measurement data given to you at the beginning of the problem*. Use this data to the full degree of accuracy in which it is presented to you. Also, *don't round quantities during a calculation*. *Wait until the end* of the calculation to do any rounding that might be appropriate. The reason for this rule is that any rounding done during a calculation introduces unnecessary and avoidable uncertainty.

Calculation Rule 2: The output of a calculation can never be more accurate than the least accurate input. Therefore, at the end of a calculation, you should round the answer so that it is no more accurate than the least accurate piece of data given to you at the beginning of the calculation.

Calculation Rule 2 is easy to apply if all the data given to you at the beginning of a calculation is expressed in the same units to the same degree of accuracy (say tenths of a centimeter). Then at the end of the calculation the output should also be rounded to the same degree of accuracy (tenths of a centimeter).

Generally, in calculations that involve only addition and subtraction, all inputs will be expressed in the same units (say centimeters), but some of the degrees of accuracy of various inputs may differ. (For example, some inputs may be given to the nearest tenth of a centimeter while others are given to the nearest hundredth of a centimeter.) In this situation, at the end of the calculations, round the answer to the degree of accuracy of the least accurate piece of input data (in this case, tenths of centimeters).

Often, calculations involving multiplication and division have input data expressed in different units to different degrees of accuracy. There is a technique that is frequently used to control uncertainty in calculations with input data that is given in mixed units with mixed degrees of accuracy. The idea behind this technique is to keep track of the *numbers of significant figures* in each measurement. The **numbers of significant figures** in a measurement is the number of digits in the measurement that are known to some degree of reliability.[4]

The following table gives examples of the number of significant figures in various measurements. The student should study these examples to learn how to count the number of significant figures in a measurement. (No additional instruction about how to count significant figures will be presented.)

---

[4] Furthermore, information about significant figures can be found online at
http://www.chem.sc.edu/morgan/resources/sigfigs/
and
http://chemed.chem.purdue.edu/genchem/topicreview/bp/chl/sigfigs.html.

Measurement	Number of Significant Figures
324.7 cm to the nearest tenth of a centimeter	4
1289.30 cm to the nearest hundredth of a cm	6
5280 feet to the nearest foot	4
5280 feet to the nearest 10 feet	3
93,000,000 miles to the nearest million miles	2
93,000,000 miles to the nearest 10,000 miles	4
.0052 in to the nearest ten thousandth of an inch	2
.0350 meters to the nearest ten thousandth of a meter	3

The examples in the last two lines of the table show that in a measurement with value less than 1, the zeroes to the left of the left-most non-zero digit are not counted as significant figures.

We remark that there are some numbers called *exact numbers* which appear in calculations that have *infinitely many* significant figures. For example, numbers that are used to convert units like “5280 feet in a mile” or “12 eggs in a dozen” are exact numbers. Numerical constants like

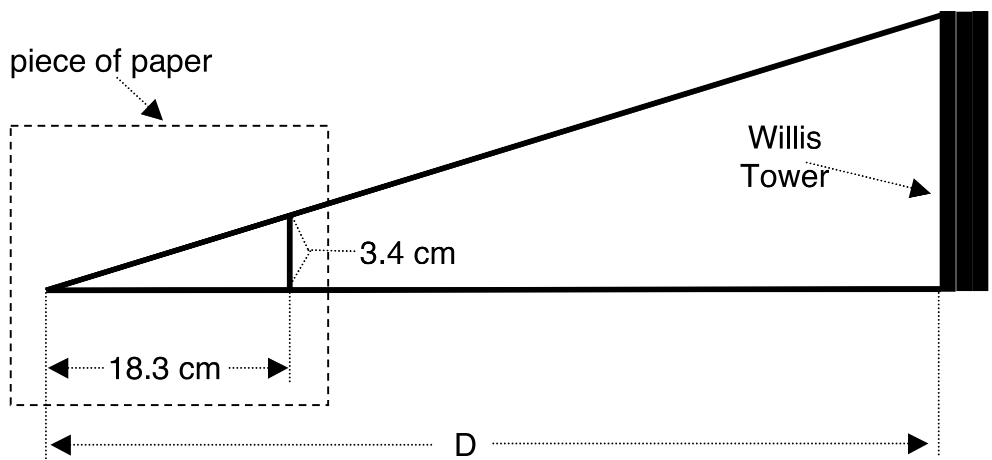
$$\sqrt{2} = 1.414213562\ldots \quad \text{and} \quad \pi = 3.141592654\ldots$$

have as many significant figures as you can determine by using your pocket calculator or by looking up the value in a book or online.

The appropriate situation in which to use significant figures is in a calculation that involves *multiplication* or *division* in which the input data is given in *different units* to different degrees of accuracy. The appropriate time to use significant figures is at the very end of such a calculation when it is time to round the final answer. At that point you should round the final answer so that it has the same number of significant figures as the piece of input data with the fewest significant figures. (Any *exact numbers* appearing in the calculation should not be used to limit the accuracy of the final answer because they have infinitely many significant figures.)

The use of significant figures in calculations involving multiplication or division is an imperfect technique. It is a “rule of thumb” that avoids some of the inaccuracy that can naturally enter such calculations, but it is far from flawless. Using significant figures to round at the end of a calculation can lead to answers that are less than optimally accurate; however, it would require an application of interval analysis to reveal this inaccuracy, and we are not going to pursue that topic further.

Here is an example of the use of the *Calculation Rules* in a measurement. The Willis Tower (formerly known as the Sears Tower) in Chicago is 1451 feet high to the nearest foot. Suppose you are standing some unknown distance D away from the Willis Tower and you wish to measure D in miles. By sighting the Willis Tower you are able to draw a right triangle on a piece of paper where height and base are proportional to the height of the Willis Tower and your distance D from the Willis Tower, respectively. Suppose the height and base of this right triangle are 3.4 cm and 18.3 cm to the nearest tenth of a centimeter, respectively. (This situation is illustrated in the following figure.)



The two right triangles in this figure are *similar*. (We will review this concept in Lesson 2.) Hence, the equation

$$\frac{D}{1451} = \frac{18.3}{3.4}$$

expresses D in feet. Solving this equation we get

$$D = 1431 \text{ x } \frac{18.3}{3.4} = 7809.794118\ldots \text{ in feet.}$$

(Don’t round yet. Obey Calculation Rule 1.)

$$\text{In miles } D = \frac{7809.794118}{5280} = 1.479127674\ldots$$

Now, we use Calculation Rule 2. The piece of input data with the fewest significant figures is 3.4. It has 2 significant figures. So we round D to have 2 significant figures.

D = 1.5 miles to the nearest tenth of a mile.

Homework Problem 1. The statement "the average distance from the Earth to the Sun is 92,960,000 miles" can be found on the internet. Which of the following statements is true?

a) The average distance from the Earth to the Sun is 93,000,000 to the nearest 1,000,000 miles.

b) The average distance from the Earth to the Sun is 93,000,000 to the nearest 100,000 miles.

c) The average distance from the Earth to the Sun is 93,000,000 to the nearest 10,000 miles.

d) The average distance from the Earth to the Sun is 93,000,000 to the nearest 1000 miles.

Hint: Turn each of the statements **a)** through **d)** into an inequality that gives a range of values for the distance from the Earth to the Sun.

Homework Problem 2. Suppose that a person measures the length L of a line segment with a ruler and then fills in the blanks in one of the statements 1) through 4) or in diagram 5) below. Given this information, your job is to fill in all the other blanks consistently. Each of problems **a)** through **f)** on the next three pages poses an exercise of this type.

1) The ruler is graduated in units of ______________________________

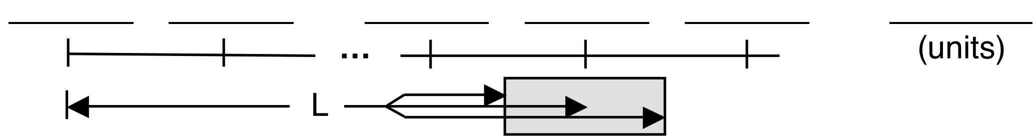
and we find that L is closer to the _____________ mark on the ruler than it is to either the _____________ mark or the _____________ mark.

2) L = ________________ to the nearest ________________________.

3) ________________ ≤ L ≤ ________________.

4) L = ________________ ± ________________________.

5)



a) Suppose that statement **4)** reads "L = .094 ± .0005 m." Fill in the blanks in statements **1)**, **2)** and **3)** and in diagram **5)** to make them have the same meaning as statement **4).**

1) The ruler is graduated in units of ______________________________

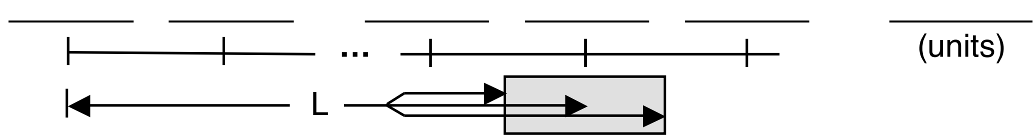
and we find that L is closer to the ______________ mark on the ruler than it is to

either the ______________ mark or the ______________ mark.

2) L = _________________ to the nearest _________________________.

3) _________________ ≤ L ≤ _________________.

5)



b) Suppose statement **2)** reads "L = 6530 miles to the nearest 10 miles". Fill in the blanks in statements **1)**, **3)** and **4)** and in diagram **5)** to make then have the same meaning as statement **2)**.

1) The ruler is graduated in units of ______________________________

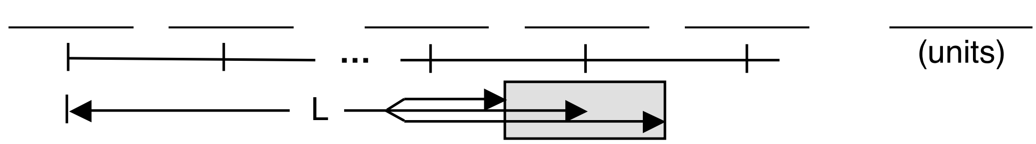
and we find that L is closer to the ______________ mark on the ruler than it is to

either the ______________ mark or the ______________ mark.

3) _________________ ≤ L ≤ _________________.

4) L = _________________ ± _________________________.

5)



c) Suppose statement **2)** reads "L = 25 inches to the nearest quarter of an inch". Fill in the blanks in statements **1), 3)** and **4)** and in diagram **5)** to make then have the same meaning as statement **2)**.

1) The ruler is graduated in units of ______________________________

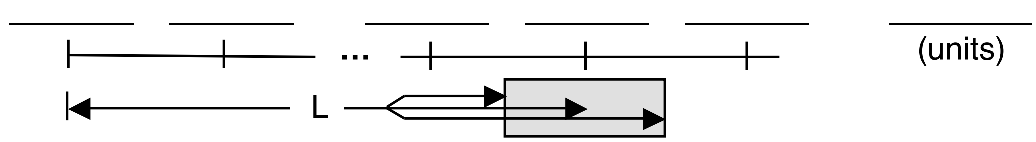
and we find that L is closer to the ____________ mark on the ruler than it is to

either the ____________ mark or the ____________ mark.

3) ________________ ≤ L ≤ ________________.

4) L = ________________ ± ________________________.

5)



d) Suppose statement **3)** reads "7950 m ≤ L ≤ 8050 m". Fill in the blanks in statements **1), 2)** and **3)** and in diagram **5)** to make them have the same meaning as statement **4)**. (1 *nanometer* = 1 billionth of a meter.)

1) The ruler is graduated in units of ______________________________

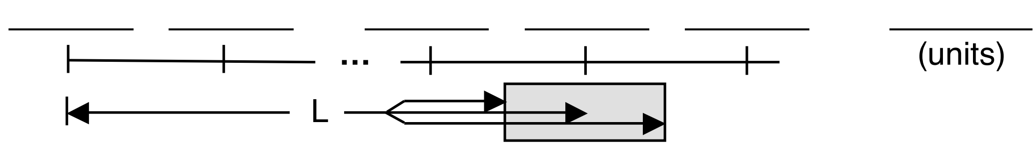
and we find that L is closer to the ____________ mark on the ruler than it is to

either the ____________ mark or the ____________ mark.

2) L = ________________ to the nearest ________________________.

4) L = ________________ ± ________________________.

5)



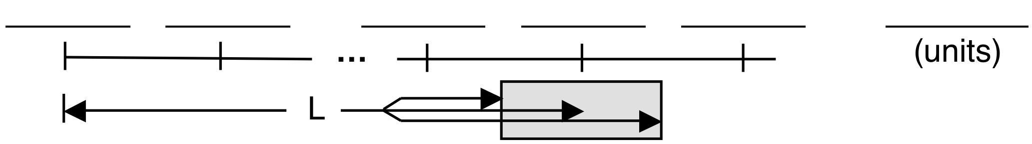
e) Suppose statement **1)** reads “The ruler is graduated in units of tenths of a kilometer and we find that L is closer to the 8.3 km mark than it is to either the 8.2 km mark or the 8.4 km mark.” Fill in the blanks in statements **2)**, **3)** and **4)** and in diagram **5)** to make then have the same meaning as statement **1)**.

2) L = __________________ to the nearest ___________________________.

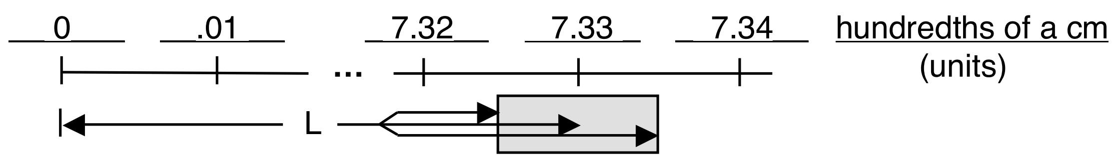
3) __________________ ≤ L ≤ __________________.

4) L = __________________ ± ___________________________.

5)



f) Suppose diagram **5)** reads



Fill in the blanks in statements **1), 2), 3)** and **4)** to make them have the same meaning as diagram **5)**.

1) The ruler is graduated in units of _________________________________

and we find that L is closer to the _______________ mark on the ruler than it is to either the _______________ mark or the _______________ mark.

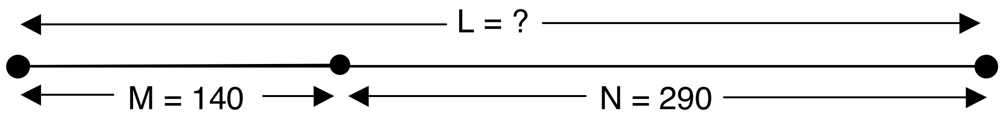
2) L = __________________ to the nearest ___________________________.

3) __________________ ≤ L ≤ __________________.

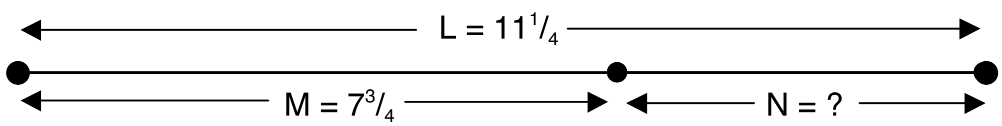
4) L = __________________ ± ___________________________.

Homework Problem 3. This problem asks you to calculate the range of possible values of an unknown length or area. So the final answer will be an inequality that gives lower and upper bounds on the value of the length or area you have calculated.

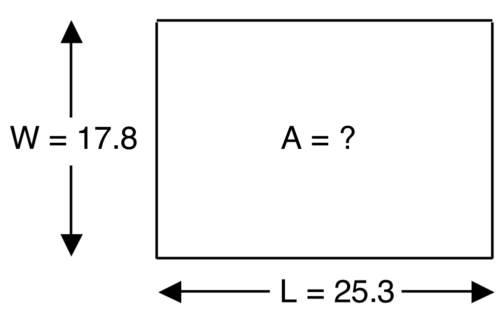
a) Suppose that a line segment of unknown length L is divided into two smaller line segments of known lengths M = 140 cm and N = 290 cm to the nearest 10 cm. Find the range of possible values of L.



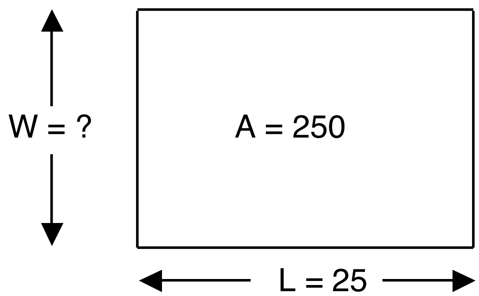
b) Suppose that a line segment of length $L = 11^{1}/_{4}$ in to the nearest quarter of an inch is divided into two smaller line segments. One of the smaller line segments is of length $M = 7^{3}/_{4}$ in to the nearest quarter of an inch, and the other is of unknown length N. Find the range of possible values of N.



c) Suppose that a rectangle has length L = 25.3 cm and width W = 17.8 cm to the nearest tenth of a centimeter. Find the range of possible values of the area $A = L \times W$ of this rectangle in square centimeters. Calculate the width of this range of values.



d) Suppose that a rectangle has length L = 25 meters to the nearest meter and area 250 square meters to the nearest square meter. It's width W is unknown. Find the range of possible values of the width W of this rectangle. Calculate the width of this range of values.



Homework Problem 4. Determine the number of significant figures in each of the following six measurements.

a) .00820 miles to the nearest hundred thousandth of a mile.

b) 307.40 cm to the nearest tenth of a centimeter.

c) 8,520,000 km to the nearest 100 kilometers.

d) 400.0030 m to the nearest thousandth of a meter.

e) $\frac{1}{3}$ = .333333…