ELSEVIER

Topology and its Applications 71 (1996) 277–293

TOPOLOGY AND ITS APPLICATIONS

Mapping swirls and pseudo-spines of compact 4-manifolds

Fredric D. Ancel *, Craig R. Guilbault [1]

Department of Mathematical Sciences, University of Wisconsin-Milwaukee, Milwaukee, WI 53201, USA

Received 14 September 1995; revised 23 October 1995

Abstract

A compact subset X of the interior of a compact manifold M is a *pseudo-spine* of M if $M - X$ is homeomorphic to $(\partial M) \times [0, \infty)$. It is proved that a 4-manifold obtained by attaching k essential 2-handles to a $B^3 \times S^1$ has a pseudo-spine which is obtained by attaching k B^2's to an S^1 by maps of the form $z \mapsto z^n$. This result recovers the fact that the Mazur 4-manifold has a disk pseudo-spine (which may then be shrunk to an arc). To prove this result, the *mapping swirl* (a "swirled" mapping cylinder) of a map to a circle is introduced, and a fundamental property of mapping swirls is established: homotopic maps to a circle have homeomorphic mapping swirls.

Several conjectures concerning the existence of pseudo-spines in compact 4-manifolds are stated and discussed, including the following two related conjectures: every compact contractible 4-manifold has an arc pseudo-spine, and every compact contractible 4-manifold has a handlebody decomposition with no 3- or 4-handles. It is proved that an important class of compact contractible 4-manifolds described by Poénaru satisfies the latter conjecture.

Keywords: Pseudo-spine; Mazur 4-manifold; Mapping swirl; Poénaru 4-manifolds

AMS classification: 57N13

1. Introduction

A compact subset X of the interior of a compact manifold M is a called a (*topological*) *spine* of M if M is homeomorphic to the mapping cylinder of a map from ∂M to X. X is called a *pseudo-spine* of M if $M - X$ is homeomorphic to $(\partial M) \times [0, \infty)$.

It is proved in [1] that for $n \geqslant 5$, every compact contractible n-manifold has a wild arc spine. It is observed, however, that in general compact contractible 4-manifolds don't have arc spines. In fact, a compact contractible 4-manifold with an arc spine must be

* Corresponding author. E-mail: ancel@csd.uwm.edu.

[1] E-mail: craigg@csd.uwm.edu.

PII S0166-8641(96)00007-7

either a 4-ball or the cone over a nontrivial homotopy 3-sphere (if one exists). Thus, a compact contractible 4-manifold with a nonsimply connected boundary can't have an arc spine.

The Mazur 4-manifold [6] is a compact contractible 4-manifold with a nonsimply connected boundary. It is a well-known consequence of [5,3] that the Mazur 4-manifold has an arc pseudo-spine.

The naively optimistic conjecture motivating this paper is: every compact contractible 4-manifold has an arc pseudo-spine. The mathematical content of the paper arises from the introduction of the *mapping swirl* construction which allows us to reinterpret and generalize the method of [5]. In Section 2 of this article the *mapping swirl* of a map to S^1 is defined and two fundamental theorems about it are proved: Theorem 1: Homotopic maps from a compact metric space to S^1 have homeomorphic mapping swirls. Theorem 2: For a compact metric space X and an integer $n \neq 0$, the mapping swirl and the mapping cylinder of the map $(x,z) \mapsto z^n : X \times S^1 \to S^1$ are homeomorphic. Section 3 applies these theorems to produce simple pseudo-spines for the special class of 4-manifolds obtained by adding finitely many essential 2-handles to $B^3 \times S^1$. This approach recovers the previously known result that Mazur's compact contractible 4-manifold has an arc pseudo-spine. Section 4 speculates about the possibility of finding simple pseudo-spines for all compact 4-manifolds. In particular, it includes the conjecture that every compact contractible 4-manifold has an arc pseudo-spine. It also states a closely related conjecture: every compact contractible 4-manifold has a handlebody decomposition with no 3- or 4-handles. It then presents a proof that this conjecture holds for an important class of compact contractible 4-manifolds described by Poénaru.

2. Mapping swirls

Let $f : X \to S^1$ be a map from a compact metric space X to S^1. Intuitively, the *mapping swirl* of f is obtained from the mapping cylinder of f by "swirling" the fibers of the mapping cylinder around infinitely many times in the S^1-direction as they approach the S^1-end of the mapping cylinder. To make this informal definition precise, we use the fact that the mapping cylinder of f embeds naturally in $(CX) \times S^1$, where CX is the cone on X. The mapping swirl of f is defined as a subset of $(CX) \times S^1$. The swirling effect is achieved by using the S^1-factor. We also define the *double mapping swirl* of f, by "swirling" the fibers of the double mapping cylinder of f at both ends. The double mapping swirl of f is defined as a subset of $(\Sigma X) \times S^1$, where ΣX is the suspension of X. The S^1-factor is again used to achieve the swirling effect.

Simple examples show that two maps from a compact metric space to S^1 may differ by only a slight homotopy and yet have nonhomeomorphic mapping cylinders and double mapping cylinders. In contrast, our principal result, Theorem 1, says that homotopic maps from a compact metric space to S^1 have homeomorphic mapping swirls. The swirling process kills the topological difference between the mapping cylinders of homotopic maps.

Definition. Let X be a compact metric space. The *suspension* of X, denoted ΣX, is the quotient space $[-\infty,\infty] \times X/\{\{-\infty\} \times X,\ \{\infty\} \times X\}$. Let $q:[-\infty,\infty] \times X \to \Sigma X$ denote the quotient map. For $(t,x) \in [-\infty,\infty] \times X$, let $tx = q((t,x))$; and let $\pm\infty = q(\{\pm\infty\} \times X)$. The *cone* on X, denoted CX, is $q([0,\infty] \times X)$.

Definition. Let $f:X \to Y$ be a map between compact metric spaces. The *mapping cylinder* of f, denoted Cyl(f), is the subspace

$$\big\{\big(tx, f(x)\big) \in (CX) \times Y:\ (t,x) \in [0,\infty) \times X\big\} \cup \big(\{\infty\} \times Y\big)$$

of $(CX) \times Y$. The *double mapping cylinder* of f, denoted DblCyl(f), is the subspace

$$\big\{\big(tx, f(x)\big) \in (\Sigma X) \times Y:\ (t,x) \in (-\infty,\infty) \times X\big\} \cup \big(\{-\infty,\infty\} \times Y\big)$$

of $(\Sigma X) \times Y$. For $-\infty < t < \infty$, call the set $\{(tx, f(x)):\ x \in X\}$ the *t-level* of DblCyl(f); it is homeomorphic to X. Call $\{\pm\infty\} \times Y$ the $\pm\infty$*-level* of DblCyl(f). Observe that the union of the t-levels of DblCyl(f) for $0 \leqslant t \leqslant \infty$ is precisely Cyl(f).

To reconcile this definition of the mapping cylinder of f with the usual definition, consider the map from the disjoint union $([0,\infty] \times X) \cup Y$ onto the subset of $(CX) \times Y$ which we have called Cyl(f) which sends $(t,x) \in ([0,\infty] \times X)$ to $(tx, f(x))$ and sends $y \in Y$ to (∞, y). The set of inverse images of this map determines the decomposition of $([0,\infty] \times X) \cup Y$ in which the only nonsingleton elements are sets of the form $(f^{-1}(y) \times \{\infty\}) \cup \{y\}$ for $y \in Y$. This is exactly the decomposition which is determined by the inverse images of the "usual" quotient map from $([0,\infty] \times X) \cup Y$ to the "usual" mapping cylinder of f. Consequently, Cyl(f) is homeomorphic to the "usual" mapping cylinder of f. Similarly, DblCyl(f) is homeomorphic to the "usual" double mapping cylinder of f.

Definition. Let X be a compact metric space and let $f:X \to S^1$ be a map. The *mapping swirl* of f, denoted Swl(f), is the subspace

$$\big\{\big(tx, \mathrm{e}^{2\pi \mathrm{i} t} f(x)\big) \in (CX) \times S^1:\ (t,x) \in [0,\infty) \times X\big\} \cup \big(\{\infty\} \times S^1\big)$$

of $(CX) \times S^1$. The *double mapping swirl* of f, denoted DblSwl(f), is the subspace

$$\big\{\big(tx, \mathrm{e}^{2\pi \mathrm{i} t} f(x)\big) \in (\Sigma X) \times S^1:\ (t,x) \in (-\infty,\infty) \times X\big\} \cup \big(\{-\infty,\infty\} \times S^1\big)$$

of $(\Sigma X) \times S^1$. For $-\infty < t < \infty$, call the set $\{(tx, \mathrm{e}^{2\pi \mathrm{i} t} f(x)):\ x \in X\}$ the *t-level* of DblSwl(f); it is homeomorphic to X. Call $\{\pm\infty\} \times S^1$ the $(\pm\infty)$*-level* of DblSwl(f). Observe that the union of the t-levels of DblSwl(f) for $0 \leqslant t \leqslant \infty$ is precisely Swl(f). For $x \in X$, call the set $\{(tx, \mathrm{e}^{2\pi \mathrm{i} t} f(x)):\ -\infty < t < \infty\}$ the *x-fiber* of DblSwl(f), and call the set $\{(tx, \mathrm{e}^{2\pi \mathrm{i} t} f(x)):\ 0 \leqslant t < \infty\}$ the *x-fiber* of Swl(f). If $g:X \to S^1$ is another map and $x \in X$, then the x-fiber of Swl(f) (DblSwl(f)) and the x-fiber of Swl(g) (DblSwl(g)) are called *corresponding fibers*.

Theorem 1. *If X is a compact metric space, and $f, g:X \to S^1$ are homotopic maps, then* Swl(f) *is homeomorphic to* Swl(g). *Furthermore, the homeomorphism maps the*

0- *and* ∞-*levels of* Swl(f) *onto the* 0- *and* ∞-*levels of* Swl(g), *respectively, and maps each fiber of* Swl(f) *onto the corresponding fiber of* Swl(g).

Proof. The proof has two steps. First we find a homeomorphism of $(\Sigma X) \times S^1$ which carries DblSwl(f) onto DblSwl(g). This homeomorphism moves Swl(f) into DblSwl(g), because Swl$(f) \subset$ DblSwl(f). Second we find a homeomorphism of $(\Sigma X) \times S^1$ which "twists" the image of Swl(f) onto Swl(g) within DblSwl(g).

Step 1. For each $x \in X$, set $\mathcal{F}(x) = \{(tx, e^{2\pi it} f(x))\colon -\infty < t < \infty\}$ and set $\mathcal{G}(x) = \{(tx, e^{2\pi it} g(x))\colon -\infty < t < \infty\}$. $\mathcal{F}(x)$ and $\mathcal{G}(x)$ are the x-fibers of DblSwl(f) and DblSwl(g), respectively. Both lie in $((-\infty,\infty)x) \times S^1 \subset (\Sigma(X)) \times S^1$.

For each $x \in X$, the x-fibers $\mathcal{F}(x)$ and $\mathcal{G}(x)$ form a "double helix" in the cylinder $((-\infty,\infty)x) \times S^1$. The angle $\theta(x)$ between $\mathcal{F}(x)$ and $\mathcal{G}(x)$ in the S^1-direction is precisely the angle between $f(x)$ and $g(x)$ in S^1, and a twist of the cylinder $((-\infty,\infty)x) \times S^1$ in the S^1-direction through the angle $\theta(x)$ would move $\mathcal{F}(x)$ to $\mathcal{G}(x)$. Unfortunately, one can't form the "union" of these twists over all the cylinders $((-\infty,\infty)x) \times S^1$ to move DblSwl(f) to DblSwl(g) in $(\Sigma X) \times S^1$, because $\theta(x)$ may vary with x, so that there is no single rotation of $\{-\infty,\infty\} \times S^1$ that extends the twists of all the cylinders. Instead of using a twist, one observes that the helix $\mathcal{F}(x)$ can be moved to the helix $\mathcal{G}(x)$ by a slide of the cylinder $((-\infty,\infty)x) \times S^1$ in the $(-\infty,\infty)x$-direction. The length of the slide in the $(-\infty,\infty)x$-direction varies with x and is essentially determined by lifting the homotopy joining f to g in S^1 to a homotopy in $(-\infty,\infty)$. Unlike the previously considered twist, this slide extends to $\{-\infty,\infty\} \times S^1$ via the identity. This is because the slide makes no motion in the S^1-direction and preserves the "ends" of $(-\infty,\infty)x$. The details follow.

Suppose $h\colon X \times [0,1] \to S^1$ is a homotopy such that $h(x,0) = g(x)$ and $h(x,1) = f(x)$. We exploit the fact that S^1 is a group under complex multiplication to define the map $k\colon X \times [0,1] \to S^1$ by $k(x,t) = h(x,t)/h(x,0)$. Thus, $k(x,0) = 1$ and $k(x,1)g(x) = f(x)$ for $x \in X$. Let $e\colon (-\infty,\infty) \to S^1$ denote the exponential covering map $e(t) = e^{2\pi it}$. Let $\tilde{k}\colon X \times [0,1] \to (-\infty,\infty)$ be the lift of k (i.e., $e \circ \tilde{k} = k$) such that $\tilde{k}(x,0) = 0$ for all $x \in X$. Define $\sigma\colon X \to (-\infty,\infty)$ by $\sigma(x) = \tilde{k}(x,1)$. Observe that for each $x \in X$, $f(x)/e^{2\pi i\sigma(x)} = f(x)/e(\tilde{k}(x,1)) = f(x)/k(x,1) = g(x)$. Since X is compact, there is a $b \in (0,\infty)$ such that $\sigma(X) \subset (-b,b)$. As we will see, $\sigma(x)$ specifies the length of the slide of the cylinder $((-\infty,\infty)x) \times S^1$ in the $(-\infty,\infty)x$-direction that moves $\mathcal{F}(x)$ to $\mathcal{G}(x)$.

Now define the function $\Phi\colon (\Sigma X) \times S^1 \to (\Sigma X) \times S^1$ by setting $\Phi(tx, z) = ((t + \sigma(x))x, z)$ for $(t,x) \in (-\infty,\infty) \times X$ and $z \in S^1$, and by requiring that $\Phi|\{-\infty,\infty\} \times S^1 = \mathrm{id}$. Clearly Φ is continuous at each point of $(\Sigma X) \times S^1 - \{-\infty,\infty\} \times S^1$. For each $z \in S^1$, the continuity of Φ at the points $(\pm\infty, z)$ follows from the inclusions

$$\Phi\big(([t,\infty]x) \times \{z\}\big) \subset \big((t-b,\infty]x\big) \times \{z\},$$
$$\Phi\big(([-\infty,t]x) \times \{z\}\big) \subset \big([-\infty,t+b)x\big) \times \{z\}.$$

Next we verify that $\Phi($DblSwl$(f)) \subset$ DblSwl(g). To this end, let $x \in X$ and consider a typical point $(tx, e^{2\pi it} f(x))$ of the fiber $\mathcal{F}(x)$. Φ moves this point to the point

$$\begin{aligned}\big((t+\sigma(x))x, \mathrm{e}^{2\pi\mathrm{i}t}f(x)\big) &= \big((t+\sigma(x))x, \mathrm{e}^{2\pi\mathrm{i}(t+\sigma(x))}f(x)/\mathrm{e}^{2\pi\mathrm{i}\sigma(x)}\big)\\ &= \big((t+\sigma(x))x, \mathrm{e}^{2\pi\mathrm{i}(t+\sigma(x))}g(x)\big)\end{aligned}$$

which is a point of the fiber $\mathcal{G}(x)$. Consequently, $\Phi(\mathcal{F}(x)) \subset \mathcal{G}(x)$. Thus, Φ maps each fiber of DblSwl(f) into the corresponding fiber of DblSwl(g). Also $\Phi(\{-\infty,\infty\}\times S^1) = \{-\infty,\infty\}\times S^1$. Since DblSwl$(f)$ and DblSwl(g) are the unions of their fibers and of $\{-\infty,\infty\}\times S^1$, we conclude that $\Phi(\text{DblSwl}(f)) \subset \text{DblSwl}(g)$.

To complete Step 1, we must verify that Φ is a homeomorphism and that $\Phi(\text{DblSwl}(f)) = \text{DblSwl}(g)$. We accomplish this by defining the function $\overline{\Phi}: (\Sigma X)\times S^1 \to (\Sigma X)\times S^1$ by setting $\overline{\Phi}(tx, z) = ((t-\sigma(x))x, z)$ for $(t,x)\in(-\infty,\infty)\times X$ and $z\in S^1$, and by requiring that $\overline{\Phi}|\{-\infty,\infty\}\times S^1 = \text{id}$. Arguments similar to those just given show that $\overline{\Phi}$ is continuous and that $\overline{\Phi}(\text{DblSwl}(g)) \subset \text{DblSwl}(f)$. Also it is easily checked that the composition of Φ and $\overline{\Phi}$ in either order is the identity. Hence, Φ is a homeomorphism, and $\Phi(\text{DblSwl}(f)) \supset \Phi(\overline{\Phi}(\text{DblSwl}(g))) = \text{DblSwl}(g)$. So $\Phi(\text{DblSwl}(f)) = \text{DblSwl}(g)$.

Step 2. Here we will find a homeomorphism Ψ of $(\Sigma X)\times S^1$ such that

$$\Psi\big(\Phi(\text{Swl}(f))\big) = \text{Swl}(g).$$

For each $x\in X$, set $\mathcal{F}^+(x) = \{(tx, \mathrm{e}^{2\pi\mathrm{i}t}f(x)) \colon 0\leqslant t<\infty\}$ and set $\mathcal{G}^+(x) = \{(tx, \mathrm{e}^{2\pi\mathrm{i}t}g(x))\colon 0\leqslant t<\infty\}$. $\mathcal{F}^+(x)$ and $\mathcal{G}^+(x)$ are the x-fibers of Swl(f) and Swl(g), respectively.

For $x\in X$, since $\mathcal{F}^+(x)\subset\mathcal{F}(x)$, then $\Phi(\mathcal{F}^+(x))\subset\Phi(\mathcal{F}(x))\subset\mathcal{G}(x)$; also $\mathcal{G}^+(x)\subset\mathcal{G}(x)$. We will describe a homeomorphism Ψ which gives the cylinder $((-\infty,\infty)x)\times S^1$ a screw motion that carries the fiber $\mathcal{G}(x)$ onto itself and moves $\Phi(\mathcal{F}^+(x))$ onto $\mathcal{G}^+(x)$. Also Ψ will restrict to the identity on a neighborhood of $\{-\infty,\infty\}\times S^1$.

Recall that $b\in(0,\infty)$ such that $\sigma(X)\subset(-b,b)$. There is a map $\tau:(-\infty,\infty)\times X\to(-\infty,\infty)$ such that for each $x\in X$, $t\mapsto\tau(t,x):(-\infty,\infty)\to(-\infty,\infty)$ is an order preserving piecewise linear homeomorphism which restricts to the identity on $(-\infty,-b]\cup[b,\infty)$ and which moves $\sigma(x)$ to 0. For example, τ can be defined by the formulas:

$$\begin{aligned}&\tau(t,x) = \big(b/\big(b+\sigma(x)\big)\big)\big(t-\sigma(x)\big) && \text{for } t\in\big[-b,\sigma(x)\big],\\ &\tau(t,x) = \big(b/\big(b-\sigma(x)\big)\big)\big(t-\sigma(x)\big) && \text{for } t\in\big[\sigma(x),b\big],\\ &\tau(t,x) = t && \text{for } t\in(-\infty,-b]\cup[b,\infty).\end{aligned}$$

Now define the function $\Psi:(\Sigma X)\times S^1\to(\Sigma X)\times S^1$ by setting

$$\Psi(tx,z) = \big(\tau(t,x)x, \mathrm{e}^{2\pi\mathrm{i}(\tau(t,x)-t)}z\big)\quad\text{for } (t,x)\in(-\infty,\infty)\times X \text{ and } z\in S^1,$$

and by requiring that $\Psi|\{-\infty,\infty\}\times S^1 = \text{id}$. Since τ is continuous, then Ψ is continuous at each point of $(\Sigma X)\times S^1 - \{-\infty,\infty\}\times S^1$. Also since $\tau(t,x) = t$ for $t\in(-\infty,-b]\cup[b,\infty)$, then Ψ restricts to the identity on the neighborhood of $\{-\infty,\infty\}\times S^1$ in $(\Sigma X)\times S^1$ consisting of all points of the form (tx,z) where $t\in[-\infty,-b]\cup[b,\infty]$, $x\in X$ and $z\in S^1$. Hence, Ψ is continuous at each point of $\{-\infty,\infty\}\times S^1$.

Next we verify that $\Psi(\Phi(\mathrm{Swl}(f))) \subset \mathrm{Swl}(g)$. To this end, let $x \in X$ and consider a typical point p of $\mathcal{F}^+(x)$. p has the form $(tx, \mathrm{e}^{2\pi\mathrm{i}t} f(x))$ where $0 \leqslant t < \infty$. Thus $\Phi(p) = ((t + \sigma(x))x, \mathrm{e}^{2\pi\mathrm{i}(t+\sigma(x))} g(x))$. Hence,

$$\begin{aligned}\Psi\big(\Phi(p)\big) &= \big(\tau\big(t+\sigma(x), x\big)x, \mathrm{e}^{2\pi\mathrm{i}(\tau(t+\sigma(x),x)-t-\sigma(x))}\mathrm{e}^{2\pi\mathrm{i}(t+\sigma(x))}g(x)\big)\\ &= \big(\tau\big(t+\sigma(x), x\big)x, \mathrm{e}^{2\pi\mathrm{i}\tau(t+\sigma(x),x)}g(x)\big).\end{aligned}$$

Since $u \mapsto \tau(u, x): (-\infty, \infty) \to (-\infty, \infty)$ is an order preserving homeomorphism, $\tau(\sigma(x), x) = 0$ and $t \geqslant 0$, then $\tau(t + \sigma(x), x) \geqslant 0$. It follows that $\Psi(\Phi(p))$ belongs to the fiber $\mathcal{G}^+(x)$. This proves $\Psi(\Phi(\mathcal{F}^+(x))) \subset \mathcal{G}^+(x)$. So $\Psi \circ \Phi$ maps each fiber of $\mathrm{Swl}(f)$ into the corresponding fiber of $\mathrm{Swl}(g)$. Also $\Psi \circ \Phi(\{\infty\} \times S^1) = \{\infty\} \times S^1$. Since $\mathrm{Swl}(f)$ and $\mathrm{Swl}(g)$ are the unions of their fibers and of $\{\infty\} \times S^1$, we conclude that $\Psi(\Phi(\mathrm{Swl}(f))) \subset \mathrm{Swl}(g)$.

It remains to establish that $\Psi: (\Sigma X) \times S^1 \to (\Sigma X) \times S^1$ is a homeomorphism and that $\Psi \circ (\mathrm{Swl}(f)) = \mathrm{Swl}(g)$. To this end, first note that there is a map $\bar{\tau}: (-\infty, \infty) \times X \to (-\infty, \infty)$ such that for each $x \in X$, $t \mapsto \bar{\tau}(t, x): (-\infty, \infty) \to (-\infty, \infty)$ is the inverse of the homeomorphism $t \mapsto \tau(t, x): (-\infty, \infty) \to (-\infty, \infty)$. (Thus, for each $x \in X$, $t \mapsto \bar{\tau}(t, x): (-\infty, \infty) \to (-\infty, \infty)$ is an order preserving piecewise linear homeomorphism which restricts to the identity on $(-\infty, -b] \cup [b, \infty)$ such that $\bar{\tau}(0, x) = \sigma(x)$, and $\bar{\tau}(\tau(t, x), x) = t$ and $\tau(\bar{\tau}(t, x), x) = t$ for $-\infty < t < \infty$.) Then define the function $\overline{\Psi}: (\Sigma X) \times S^1 \to (\Sigma X) \times S^1$ by setting

$$\overline{\Psi}(tx, z) = \big(\bar{\tau}(t, x)x, \mathrm{e}^{2\pi\mathrm{i}(\bar{\tau}(t,x)-t)}z\big) \quad \text{for } (t, x) \in (-\infty, \infty) \times X \text{ and } z \in S^1,$$

and by requiring that $\overline{\Psi}|\{-\infty, \infty\} \times S^1 = \mathrm{id}$. The proof of the continuity of $\overline{\Psi}$ is similar to the proof of the continuity of Ψ.

Next we verify that $\overline{\Psi}(\mathrm{Swl}(g)) \subset \Phi(\mathrm{Swl}(f))$. To this end, let $x \in X$ and consider a typical point p of $\mathcal{G}^+(x)$. p has the form $(tx, \mathrm{e}^{2\pi\mathrm{i}t} g(x))$ where $0 \leqslant t < \infty$, and

$$\overline{\Psi}(p) = \big(\bar{\tau}(t, x)x, \mathrm{e}^{2\pi\mathrm{i}(\bar{\tau}(t,x)-t)}\mathrm{e}^{2\pi\mathrm{i}t}g(x)\big) = \big(\bar{\tau}(t, x)x, \mathrm{e}^{2\pi\mathrm{i}\bar{\tau}(t,x)}g(x)\big).$$

Since $u \mapsto \bar{\tau}(u, x): (-\infty, \infty) \to (-\infty, \infty)$ is an order preserving homeomorphism, $\bar{\tau}(0, x) = \sigma(x)$ and $t \geqslant 0$, then $\bar{\tau}(t, x) = u + \sigma(x)$ for some $u \geqslant 0$. Hence,

$$\overline{\Psi}(p) = \big(\big(u + \sigma(x)\big)x, \mathrm{e}^{2\pi\mathrm{i}(u+\sigma(x))}g(x)\big).$$

Since $u \geqslant 0$, then the point $(ux, \mathrm{e}^{2\pi\mathrm{i}u} f(x))$ belongs to $\mathcal{F}^+(x)$, and $\Phi(ux, \mathrm{e}^{2\pi\mathrm{i}u} f(x)) = ((u + \sigma(x))x, \mathrm{e}^{2\pi\mathrm{i}(u+\sigma(x))} g(x))$. Consequently,

$$\overline{\Psi}(p) = \Phi\big(ux, \mathrm{e}^{2\pi\mathrm{i}u} f(x)\big) \in \Phi\big(\mathcal{F}^+(x)\big).$$

This proves $\overline{\Psi}(\mathcal{G}^+(x)) \subset \Phi(\mathcal{F}^+(x))$. So $\overline{\Psi}$ maps each fiber of $\mathrm{Swl}(g)$ into the Φ-image of the corresponding fiber of $\mathrm{Swl}(f)$. Also $\overline{\Psi}(\{\infty\} \times S^1) = \{\infty\} \times S^1 = \Phi(\{\infty\} \times S^1)$. Since $\mathrm{Swl}(g)$ and $\mathrm{Swl}(f)$ are the unions of their fibers and of $\{\infty\} \times S^1$, we conclude that $\overline{\Psi}(\mathrm{Swl}(g)) \subset \Phi(\mathrm{Swl}(f))$.

It is easy to verify that the composition of Ψ and $\overline{\Psi}$ in either order is the identity. (Remember that $\bar{\tau}(\tau(t, x)x) = t$ and $\tau(\bar{\tau}(t, x), x) = t$ for $x \in X$ and $-\infty < t < \infty$.) Hence, Ψ is a homeomorphism.

We have seen that $\Psi(\Phi(\mathrm{Swl}(f))) \subset \mathrm{Swl}(g)$ and $\overline{\Psi}(\mathrm{Swl}(g)) \subset \Phi(\mathrm{Swl}(f))$. So

$$\Psi\big(\Phi\big(\mathrm{Swl}(f)\big)\big) \supset \Psi\big(\overline{\Psi}\big(\mathrm{Swl}(g)\big)\big) = \mathrm{Swl}(g).$$

Thus, the homeomorphism $\Psi \circ \Phi$ maps $\mathrm{Swl}(f)$ onto $\mathrm{Swl}(g)$.

In the course of the proof, we have seen that for each $x \in X$, $\Psi(\Phi(\mathcal{F}^+(x))) \subset \mathcal{G}^+(x)$ and $\overline{\Psi}(\mathcal{G}^+(x)) \subset \Phi(\mathcal{F}^+(x))$. So $\Psi(\Phi(\mathcal{F}^+(x))) \supset \Psi(\overline{\Psi}(\mathcal{G}^+(x))) = \mathcal{G}^+(x)$. Thus, $\Psi(\Phi(\mathcal{F}^+(x))) = \mathcal{G}^+(x)$. In other words, the homeomorphism $\Psi \circ \Phi$ maps each fiber of $\mathrm{Swl}(f)$ onto the corresponding fiber of $\mathrm{Swl}(g)$.

A typical point of the 0-level of $\mathrm{Swl}(f)$ has the form $(0x, f(x))$ and

$$\begin{aligned}\Psi\big(\Phi\big(0x, f(x)\big)\big) &= \Psi\big(\sigma(x)x, \mathrm{e}^{2\pi\mathrm{i}\sigma(x)}g(x)\big)\\ &= \big(\tau\big(\sigma(x), x\big)x, \mathrm{e}^{2\pi\mathrm{i}(\tau(\sigma(x),x)-\sigma(x))}\mathrm{e}^{2\pi\mathrm{i}\sigma(x)}g(x)\big) = \big(0x, g(x)\big)\end{aligned}$$

because $\tau(\sigma(x), x) = 0$. Thus, $\Psi \circ \Phi$ maps the 0-level of $\mathrm{Swl}(f)$ onto the 0-level of $\mathrm{Swl}(g)$.

Since $\Psi(\Phi(\{-\infty\} \times S^1)) = \Psi(\{-\infty\} \times S^1) = \{-\infty\} \times S^1$, then $\Psi \circ \Phi$ maps the ∞-level of $\mathrm{Swl}(f)$ onto the ∞-level of $\mathrm{Swl}(g)$. □

The next theorem and its corollaries make it possible to identify the mapping swirls of a special types of maps.

Theorem 2. *If X is a compact metric space, n is a nonzero integer, and $f: X \times S^1 \to S^1$ is the map $f(x, z) = z^n$, then $\mathrm{Cyl}(f)$ is homeomorphic to $\mathrm{Swl}(f)$. Furthermore, the homeomorphism maps the t-level of $\mathrm{Cyl}(f)$ onto the t-level of $\mathrm{Swl}(f)$ for $0 \leqslant t \leqslant \infty$.*

Proof. We will find a homeomorphism from $C(X \times S^1) \times S^1$ to itself which carries $\mathrm{Cyl}(f)$ onto $\mathrm{Swl}(f)$ by twisting motion in the S^1-direction in the $C(X \times S^1)$ factor of $C(X \times S^1) \times S^1$. This is possible because of the S^1-factor in the domain of f and the special form of f.

Define the function

$$\phi: C\big(X \times S^1\big) \to C\big(X \times S^1\big)$$

by setting $\phi(t(x, z)) = t(x, \mathrm{e}^{-2\pi\mathrm{i}t/n}z)$ for $t \in [0, \infty)$ and $(x, z) \in X \times S^1$ and $\phi(\infty) = \infty$. ϕ is clearly continuous on $C(X \times S^1) - \{\infty\}$; and because ϕ maps the t-level of $C(X \times S^1)$ into itself, then ϕ is continuous at ∞. We show that ϕ is a homeomorphism of $C(X \times S^1)$ by exhibiting its inverse. Indeed, let us define the function $\bar{\phi}: C(X \times S^1) \to C(X \times S^1)$ by setting $\bar{\phi}(t(x, z)) = t(x, \mathrm{e}^{2\pi\mathrm{i}t/n}z)$ for $t \in [0, \infty)$ and $(x, z) \in X \times S^1$ and $\bar{\phi}(\infty) = \infty$. Then $\bar{\phi}$ is continuous by an argument similar to the one just given. Also it is easily checked that the composition of ϕ and $\bar{\phi}$ in either order is the identity. So ϕ and $\bar{\phi}$ are homeomorphisms.

Next define a homeomorphism $\Phi: C(X \times S^1) \times S^1 \to C(X \times S^1) \times S^1$ by $\Phi = \phi \times \mathrm{id}$. Clearly $\overline{\Phi} = \bar{\phi} \times \mathrm{id}$ defines the homeomorphism of $C(X \times S^1) \times S^1$ which is the inverse of Φ.

We now prove that $\Phi(\text{Cyl}(f)) = \text{Swl}(f)$. Let $0 \leqslant t < \infty$, and consider a typical point $p = (t(x,z), f(x,z)) = (t(x,z), z^n)$ of the t-level of $\text{Cyl}(f)$ where $(x,z) \in X \times S^1$. Set $z' = \mathrm{e}^{-2\pi \mathrm{i} t/n} z$. Then

$$\begin{aligned}\Phi(p) &= \big(\phi\big(t(x,z), z^n\big)\big) = \big(t\big(x, \mathrm{e}^{-2\pi \mathrm{i} t/n} z\big), z^n\big) \\ &= \big(t\big(x, \mathrm{e}^{-2\pi \mathrm{i} t/n} z\big), \mathrm{e}^{2\pi \mathrm{i} t}\big(\mathrm{e}^{-2\pi \mathrm{i} t/n} z\big)^n\big) \\ &= \big(t(x,z'), \mathrm{e}^{2\pi \mathrm{i} t}(z')^n\big) = \big(t(x,z'), \mathrm{e}^{2\pi \mathrm{i} t} f(x,z')\big).\end{aligned}$$

So $\Phi(p)$ belongs to the t-level of $\text{Swl}(f)$. Also $\Phi(\{\infty\} \times S^1) = \{\infty\} \times S^1$. It follows that $\Phi(\text{Cyl}(f)) \subset \text{Swl}(f)$, and Φ maps the t-level of $\text{Cyl}(f)$ into the t-level of $\text{Swl}(f)$ for $0 \leqslant t \leqslant \infty$.

A similar argument shows that $\overline{\Phi}$ maps the t-level of $\text{Swl}(f)$ into the t-level of $\text{Cyl}(f)$ for $0 \leqslant t \leqslant \infty$. Indeed, if $0 \leqslant t < \infty$ and $p = (t(x,z)\mathrm{e}^{2\pi \mathrm{i} t} f(x,z)) = (t(x,z), \mathrm{e}^{2\pi \mathrm{i} t} z^n)$ is a typical point of the t-level of $\text{Swl}(f)$, and we set $z' = \mathrm{e}^{2\pi \mathrm{i} t/n} z$, then

$$\overline{\Phi}(p) = \big(\bar{\phi}\big(t(x,z)\mathrm{e}^{2\pi \mathrm{i} t} z^n\big)\big) = \big(t\big(x, \mathrm{e}^{2\pi \mathrm{i} t/n} z\big), \big(\mathrm{e}^{2\pi \mathrm{i} t/n} z\big)^n\big) = \big(t(x,z'), f(x,z)\big)$$

which is a point of the t-level of $\text{Cyl}(f)$. Also $\overline{\Phi}(\{\infty\} \times S^1) = \{\infty\} \times S^1$. Hence, $\overline{\Phi}(\text{Swl}(f)) \subset \text{Cyl}(f)$. Since $\overline{\Phi} = \Phi^{-1}$, it follows that $\Phi(\text{Cyl}(f)) = \text{Swl}(f)$, and Φ maps the t-level of $\text{Cyl}(f)$ onto the t-level of $\text{Swl}(f)$ for $0 \leqslant t \leqslant \infty$. □

We now exploit Theorems 1 and 2 together to state two corollaries which allows us to identify the mapping swirls of certain kinds of maps.

Corollary 1. *If X is a compact metric space, $f: X \times S^1 \to S^1$ and $g: X \times S^1 \to S^1$ are homotopic maps, and $g(x,z) = z^n$ where n is a nonzero integer, then* $\text{Swl}(f)$ *is homeomorphic to* $\text{Cyl}(g)$. *Furthermore, the homeomorphism maps the* 0- *and* ∞*-levels of* $\text{Swl}(f)$ *onto the* 0- *and* ∞*-levels of* $\text{Cyl}(g)$.

Corollary 2. *If $f: S^1 \to S^1$ is a map of degree $n \neq 0$, then* $\text{Swl}(f)$ *is homeomorphic to* $\text{Cyl}(z \mapsto z^n)$. *Furthermore, the homeomorphism maps the* 0- *and* ∞*-levels of* $\text{Swl}(f)$ *onto the* 0- *and* ∞*-levels of* $\text{Cyl}(z \mapsto z^n)$. *In particular,* $\text{Swl}(f)$ *is an annulus if $n = \pm 1$, and* $\text{Swl}(f)$ *is a Möbius strip if $n = \pm 2$.*

The last assertion of this corollary follows from the observation that the mapping cylinder of the map $z \mapsto z^n : S^1 \to S^1$ is an annulus if $n = \pm 1$, and it is a Möbius strip if $n = \pm 2$.

3. Pseudo-spines of 4-manifolds

Recall that a compact subset X of the interior of a compact manifold M is a *pseudo-spine* of M if $M - X$ is homeomorphic to $(\partial M) \times [0, \infty)$.

Let $\| \; \|$ denote the Euclidean norm on $\mathbb{R}^n$: $\|x\| = (x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2}$. Set $B^n = \{x \in \mathbb{R}^n \colon \|x\| \leqslant 1\}$ and $S^n = \{x \in \mathbb{R}^{n+1} \colon \|x\| = 1\}$.

For each integer n, let $\gamma_n : S^1 \to S^1$ denote the map $\gamma_n(z) = z^n$, and let $X(n)$ denote the adjunction space $B^2 \cup_{\gamma_n} S^1$. Thus, $X(\pm 1)$ is a 2-dimensional disk, and $X(\pm 2)$ is a projective plane. If $|n| > 2$, then $X(n)$ is a 2-dimensional polyhedron which is not a 2-manifold. For nonzero integers $n_1, n_2, \ldots, n_k$, let $X(n_1, n_2, \ldots, n_k)$ denote the adjunction space $(B^2 \times \{1, 2, \ldots, k\}) \cup_\Gamma S^1$ where $\Gamma : S^1 \times \{1, 2, \ldots, k\} \to S^1$ is the map defined by $\Gamma(z, i) = \gamma_{n_i}(z)$ for $z \in S^1$ and $1 \leqslant i \leqslant k$. Thus, $X(n_1, n_2, \ldots, n_k)$ is homeomorphic to a union of $X(n_1), X(n_2), \ldots, X(n_k)$ in which all the "edge circles" of the $X(n_i)$'s are identified with a single copy of S^1.

A simple closed curve C in the boundary of a manifold N is called *essential* if it is not homotopically trivial in ∂N. If C is essential, then any 2-handle attached to N along C is also called *essential*.

Theorem 3. *Suppose $C_1, C_2, \ldots, C_k$ are disjoint essential simple closed curves in $\partial B^3 \times S^1$, and M^4 is the 4-manifold obtained by attaching disjoint 2-handles to $B^3 \times S^1$ along $C_1, C_2, \ldots, C_k$. Let $\pi : \partial B^3 \times S^1 \to S^1$ denote the projection map. For $1 \leqslant i \leqslant k$, let n_i denote the degree of the map $\pi|C_i : C_i \to S^1$. Then M^4 has a pseudo-spine which is homeomorphic to $X(n_1, n_2, \ldots, n_k)$.*

Proof. Note that $n_i \neq 0$ because C_i is essential for $1 \leqslant i \leqslant k$. We write $M^4 = (B^3 \times S^1) \cup (H_1 \cup H_2 \cup \cdots \cup H_k)$ where H_i is the 2-handle attached to $B^3 \times S^1$ along C_i. Thus, for $1 \leqslant i \leqslant k$, there is a homeomorphism $h_i : B^2 \times B^2 \to H_i$ such that $(B^3 \times S^1) \cap H_i = h_i((\partial B^2) \times B^2) \subset \partial B^3 \times S^1$ and $h_i((\partial B^2) \times \{0\}) = C_i$. For $1 \leqslant i \leqslant k$, set $D_i = h_i(B^2 \times \{0\})$; then $\partial D_i = C_i$ and D_i is the "core disk" of H_i.

Clearly $B^3 \times S^1$ is homeomorphic to $\mathrm{Cyl}(\pi)$ by a homeomorphism that takes $\partial B^3 \times S^1$ onto the 0-level of $\mathrm{Cyl}(\pi)$. In addition, Theorem 2 provides a homeomorphism from $\mathrm{Cyl}(\pi)$ to $\mathrm{Swl}(\pi)$ which takes the 0-level of $\mathrm{Cyl}(\pi)$ to the 0-level of $\mathrm{Swl}(\pi)$. The composition of these homeomorphisms allows us to identify $B^3 \times S^1$ with $\mathrm{Swl}(\pi)$ so that $\partial B^3 \times S^1$ is identified with the 0-level of $\mathrm{Swl}(\pi)$. Thus, we can regard $C_1, C_2, \ldots, C_k$ as disjoint simple closed curves lying in the 0-level of $\mathrm{Swl}(\pi)$.

Let $1 \leqslant i \leqslant k$. Observe that $\mathrm{Swl}(\pi|C_i)$ can be naturally identified with a subset of $\mathrm{Swl}(\pi)$ so that the 0-level of $\mathrm{Swl}(\pi|C_i)$ is the subset of the 0-level of $\mathrm{Swl}(\pi)$ identified with C_i, and ∞-levels of $\mathrm{Swl}(\pi|C_i)$ and $\mathrm{Swl}(\pi)$ coincide. Since $\pi|C_i : C_i \to S^1$ is a map of degree n_i, then Corollary 3 provides a homeomorphism from $\mathrm{Swl}(\pi|C_i)$ to the mapping cylinder of the map $z \mapsto z^{n_i} : S^1 \to S^1$ which preserves 0-levels and ∞-levels. Since C_i is the 0-level of $\mathrm{Swl}(\pi|C_i)$ and $C_i = \partial D_i$, then clearly $\mathrm{Swl}(\pi|C_i) \cup D_i$ is homeomorphic to $X(n_i)$.

Set $X = \bigcup_{i=1}^k \mathrm{Swl}(\pi|C_i) \cup D_i$. Then X is a compact subset of $\mathrm{int}(M^4)$, and X is clearly homeomorphic to $X(n_1, n_2, \ldots, n_k)$.

It remains to prove that $M^4 - X$ is homeomorphic to $(\partial M^4) \times [0, \infty)$. Observe that $M^4 - X$ is the union of $\mathrm{Swl}(\pi) - \bigcup_{i=1}^k \mathrm{Swl}(\pi|C_i)$ and the sets $H_i - D_i$ for $1 \leqslant i \leqslant k$. Furthermore, $\mathrm{Swl}(\pi) - \bigcup_{i=1}^k \mathrm{Swl}(\pi|C_i)$ is the union of the fibers of $\mathrm{Swl}(\pi)$ that emanate from the points of $(\partial B^3 \times S^1) - \bigcup_{i=1}^k C_i$, and each of these fibers is homeomorphic to $[0, \infty)$. We will "extend" these fibers to fill the sets $H_i - D_i$, $1 \leqslant i \leqslant k$.

We will define a homeomorphism $G:(\partial M^4)\times[0,\infty)\to M^4-X$. To begin, there is clearly a homeomorphism $F:(\partial B^3\times S^1)\times[0,\infty)\to \mathrm{Swl}(\pi)-(\{\infty\}\times S^1)$ which takes $\{(x,z)\}\times[0,\infty)$ onto the (x,z)-fiber of $\mathrm{Swl}(\pi)$, for $(x,z)\in\partial B^3\times S^1$. Indeed, the formula $F((x,z),t)=(t(x,z),\mathrm{e}^{2\pi it}z)$ for $((x,z),t)\in(\partial B^3\times S^1)\times[0,\infty)$ determines such a homeomorphism.

For each i, $1\leqslant i\leqslant k$, set

$$A_i=h_i\big((\partial B^2)\times B^2\big)\quad\text{and}\quad B_i=h_i\big(B^2\times(\partial B^2)\big).$$

A_i is called the *attaching tube* of H_i, and B_i is called the *belt tube* of H_i. Then $A_i=\mathrm{Swl}(\pi)\cap H_i$ and

$$\partial M^4=\left((\partial B^3\times S^1)-\bigcup_{i=1}^{k}\mathrm{int}(A_i)\right)\cup\left(\bigcup_{i=1}^{k}B_i\right).$$

We set

$$G\left|\left((\partial B^3\times S^1)-\bigcup_{i=1}^{k}\mathrm{int}(A_i)\right)\times[0,\infty)\right.$$
$$=F\left|\left((\partial B^3\times S^1)-\bigcup_{i=1}^{k}\mathrm{int}(A_i)\right)\times[0,\infty).\right.$$

It remains to define $G|B_i\times[0,\infty)$ for $1\leqslant i\leqslant k$. Consider a point $p\in B_i$. Then $p=h_i(x,y)$ where $(x,y)\in B^2\times(\partial B^2)$. If $x=0$, then $G(\{p\}\times[0,\infty))$ is the "deleted radius" $h_i(\{(0,ty)\colon\ 0<t\leqslant 1\})$ of the disk $h_i(\{0\}\times B^2)$ joining the center point $h_i(0,0)$ to p. If $x\neq 0$, then $G(\{p\}\times[0,\infty))$ is the union of an arc in H_i joining the point p to a point $q\in A_i$ together with the ray $F(\{q\}\times[0,\infty))$. Moreover, the arc in H_i joining p to q is the h_i-image of the subarc of the "hyperbola" $\{(sx,ty)\colon\ st=1\}$ joining the point (x,y) to the point $(x/\|x\|,\|x\|y)$. So $q=h_i(x/\|x\|,\|x\|y)$.

The precise definition of G follows. As we stated earlier,

$$G\left|\left((\partial B^3\times S^1)-\bigcup_{i=1}^{k}\mathrm{int}(A_i)\right)\times[0,\infty)\right.$$
$$=F\left|\left((\partial B^3\times S^1)-\bigcup_{i=1}^{k}\mathrm{int}(A_i)\right)\times[0,\infty).\right.$$

Now suppose $1\leqslant i\leqslant k$, $p\in B_i$ and $p=h_i(x,y)$, where $(x,y)\in B^2\times(\partial B^2)$. If $x=0$, then

$$G(p,t)=h_i\left(0,\left(\frac{1}{t+1}\right)y\right)\quad\text{for }0\leqslant t<\infty.$$

If $x\neq 0$, then

$$G(p,t)=\begin{cases}h_i\left((t+1)x,\left(\dfrac{1}{t+1}\right)y\right), & \text{if }0\leqslant t\leqslant\dfrac{1}{\|x\|}-1,\\ F\left(h_i\left(\dfrac{x}{\|x\|},\|x\|y\right),t+1-\dfrac{1}{\|x\|}\right), & \text{if }\dfrac{1}{\|x\|}-1\leqslant t<\infty.\end{cases}$$

The following remarks are intended to further clarify the properties of G. G maps

$$\left((\partial B^3 \times S^1) - \bigcup_{i=1}^{k} \operatorname{int}(A_i)\right) \times [0,\infty)$$

onto

$$\operatorname{Swl}\left(\pi\Big|\left((\partial B^3 \times S^1) - \bigcup_{i=1}^{k} \operatorname{int}(A_i)\right)\right).$$

For $1 \leqslant i \leqslant k$, G maps

$$\left\{(p,t) \in B_i \times [0,\infty)\colon\ 0 \leqslant t \leqslant \frac{1}{\|x\|} - 1,\ p = h_i(x,y),\ (x,y) \in B^2 \times (\partial B^2)\right\}$$

onto $H_i - D_i$, and G maps

$$\left\{(p,t) \in B_i \times [0,\infty)\colon\ \frac{1}{\|x\|} - 1 \leqslant 1 < \infty,\ p = h_i(x,t),\ (x,y) \in B^2 \times (\partial B^2)\right\}$$

onto $\operatorname{Swl}(\pi|A_i - C_i)$. □

Corollary 3. *Suppose C is a simple closed curve in $(\partial B^3)\times S^1$, and $M^4 = (B^3\times S^1)\cup H$ where H is a 2-handle attached to $B^3 \times S^1$ along C. Let $\pi : B^3 \times S^1 \to S^1$ denote the projection map, and suppose that the map $\pi|C : C \to S^1$ is degree one. Then M^4 has an arc pseudo-spine.*

Proof. Theorem 3 provides M^4 with a pseudo-spine X that is homeomorphic to the 2-dimensional disk $X(1)$. According to [3], X can be "squeezed" to an arc in $\operatorname{int}(M^4)$. In other words, there is an arc A in $\operatorname{int}(M^4)$ and an onto map $f : M^4 \to M^4$ such that $f(X) = A$ and f maps $M^4 - X$ homeomorphically onto $M^4 - A$. (Interpreted literally, [3] applies only in manifolds of dimension 3. However, the methods of [3] work in manifolds of all dimensions $\geqslant 3$. This is fully explained on p. 95 of [2].) Consequently, $M^4 - A$ is homeomorphic to $\partial M^4 \times [0,\infty)$, making A an arc pseudo-spine of M^4. □

Since Mazur's compact contractible 4-manifold [6] is obtained by attaching a 2-handle to $B^3 \times S^1$ along a degree one curve, we recover the result of [5,3].

Corollary 4. *Mazur's compact contractible 4-manifold has an arc pseudo-spine.*

4. Conjectures

The results proved in this paper exhibit simple pseudo-spines for a very modest collection of 4-manifolds: those obtained by attaching essential 2-handles to $B^3 \times S^1$. The following conjectures are founded on the possibly naive hope that these results can be extended to a more general class of compact 4-manifolds.

Conjecture 1. If a compact 4-manifold with boundary is homotopy equivalent to $X(n_1, n_2, \dots, n_k)$ (where $n_1, n_2, \dots, n_k$ are nonzero integers), then it has a pseudo-spine which is homeomorphic to $X(n_1, n_2, \dots, n_k)$.

In the case of a compact contractible 4-manifold, Conjecture 1 combined with the result of [3] would yield:

Conjecture 2. Every compact contractible 4-manifold has an arc pseudo-spine.

Corollary 3 provides an arc pseudo-spine for every compact contractible 4-manifold that is obtained by attaching a 2-handle to $B^3 \times S^1$. Such a 4-manifold has a handlebody decomposition consisting of a single 0-handle, a single 1-handle and a single 2-handle. No 3- or 4-handles are needed. This suggests breaking Conjecture 2 into the following two parts.

Conjecture 2A. Every piecewise linear compact contractible 4-manifold has a handlebody decomposition with no 3- or 4-handles.

Conjecture 2B. Every compact contractible 4-manifold that has a handlebody decomposition with no 3- or 4-handles has an arc pseudo-spine.

Here is a less general and apparently more elementary question than those raised by the previous conjectures. If M^4 and N^4 are 4-manifolds with boundary, define their *boundary connected sum* $M^4 \cup_\partial N^4$ to be the adjunction space $M^4 \cup_h N^4$ where h is a homeomorphism from a collared 3-ball in ∂M^4 to a collared 3-ball in ∂N^4.

Conjecture 3. If two compact 4-manifolds have arc pseudo-spines, then so does their boundary-connected sum.

If two compact contractible 4-manifolds are each obtained by attaching a single 2-handle to $B^3 \times S^1$, then their boundary connected sum has a tree pseudo-spine which is homeomorphic to the letter "H". This is proved by using the methods of the proof of Theorem 3 and [3]. (Recall that a *tree* is a compact contractible 1-dimensional polyhedron.) This raises the question of whether a tree pseudo-spine can be simplified to an arc pseudo-spine. We can ask, more generally, whether a compact 1-dimensional polyhedral pseudo-spine be simplified to a homotopy equivalent canonical model.

Conjecture 4. If a compact 4-manifold has a tree pseudo-spine, then it has an arc pseudo-spine.

Conjecture 5. If a compact noncontractible 4-manifold has a pseudo-spine which is a compact 1-dimensional polyhedron, then it has a pseudo-spine which is a wedge of circles.

There are clear limitations on the amount to which a pseudo-spine can be simplified within its homotopy class. If a compact 4-manifold has a point pseudo-spine, then it is a cone over its boundary, which implies that its boundary is simply connected. On the other hand, there are compact contractible 4-manifolds with nonsimply connected boundaries which have arc pseudo-spines (e.g., the Mazur manifold). Clearly, the arc pseudo-spines of such manifolds can't be simplified to points.

The study of spines and pseudo-spines pursued in this paper and in [1] was partially motivated by the question of whether a compact contractible n-manifold other than the n-ball can have disjoint spines. (The existence of disjoint spines is equivalent to the existence of disjoint pseudo-spines.) In [4] it is shown that for $n \geqslant 9$, there is a large family of distinct compact contractible n-manifolds with disjoint spines. We conjecture a different situation in dimension 4.

Conjecture 6. The only compact contractible 4-manifold that has disjoint spines is the 4-ball.

We conclude with some remarks concerning Conjectures 2, 2A and 2B. The "classical" examples of compact contractible 4-manifolds include, in addition to the Mazur 4-manifold, those described by Poénaru in [7]. We will sketch the construction of Poénaru's examples, and we will explain why many Poénaru 4-manifolds have handlebody decompositions with no 3- or 4-handles. Hence, they provide some evidence for Conjecture 2A. The authors, however, do not know whether Poénaru's examples have arc pseudo-spines. These manifolds are, thus, a likely place to take up the study of Conjectures 2 and 2B.

The following discussion fits most naturally into the piecewise linear category. For this reason we identify the n-ball B^n with $[0,1]^n$ for the remainder of the paper. A locally unknotted piecewise linearly embedded 2-dimensional disk D in B^4 such that $D \cap (\partial B^4) = \partial D$ is called a *slice disk* in B^4 and ∂D is called a *slice knot* in ∂B^4. A piecewise linear simple closed curve J is ∂B^4 is called a *ribbon knot* if there is a piecewise linear map $f: B^2 \to \partial B^4$ which maps ∂B^2 onto J such that the singular set of f—

$$\{p \in \partial B^4\colon\ f^{-1}(p) \text{ contains more than one point}\}$$

—is the union of a pairwise disjoint collection of piecewise linear arcs $A_1, A_2, \ldots, A_k$ in ∂B^4 and for $1 \leqslant i \leqslant k$, $f^{-1}(A_i)$ is the union of two disjoint piecewise linear arcs A_i' and A_i'' in B^2 where $A_i' \subset \text{int}(B^2)$, $A_i'' \cap (\partial B^2) = \partial A_i''$, and f maps each of A_i' and A_i'' homeomorphically onto A_i. Clearly f can be homotoped rel ∂B^2 to a piecewise linear embedding whose image is a slice disk by pushing $f|\text{int}(B^2)$ radially into $\text{int}(B^4)$ and pushing $f|A_i'$ "deeper" than the rest of $f|\text{int}(B^2)$. The slice disk formed in this manner is called a *ribbon disk*. Thus, every ribbon knot is a slice knot. The converse assertion: every slice knot is a ribbon knot, is one of the fundamental unresolved problems of knot theory.

Poénaru's construction of a compact contractible 4-manifold begins with a slice disk D in B^4 such that ∂D is knotted in ∂B^4 and with a knotted piecewise linear simple closed curve K in the boundary of a second 4-ball $\widetilde{B}^4$. Let N be a regular neighborhood of D in B^4 such that $N \cap (\partial B^4)$ is a regular neighborhood of ∂D in ∂B^4. Set $A = \mathrm{cl}(B^4 - N) \cap N$. Then A is a solid torus (i.e., A is piecewise linearly homeomorphic to $S^1 \times B^2$), and we can think of N as a 2-handle attached to $\mathrm{cl}(B^4 - N)$ along A to yield B^4. Let T be a regular neighborhood of K in $\partial \widetilde{B}^4$. Then T is a solid torus. Let $g: T \to A$ be a piecewise linearly homeomorphism. Now define the Poénaru 4-manifold $P^4(D, K)$ to be the adjunction space $\widetilde{B}^4 \cup_g \mathrm{cl}(B^4 - N)$. We can think of $\widetilde{B}^4$ as a "knotted 2-handle" with knotted attaching tube T which is attached to $\mathrm{cl}(B^4 - N)$ by the homeomorphism $g: T \to A$ to yield $P^4(D, K)$. To see that $P^4(D, K)$ is contractible, notice that $\mathrm{cl}(B^4 - N)$ becomes contractible if the core curve of A is "killed", and attaching $\widetilde{B}^4$ to $\mathrm{cl}(B^4 - N)$ by g "kills" this curve. However, $\partial P^4(D, K)$ is not simply connected because it is the union of the two nontrivial knot complements $\mathrm{cl}(\partial B^4 - (N \cap (\partial B^4)))$ and $\mathrm{cl}(\partial B^4 - T)$. See [7] for further details.

Finally we verify that some Poénaru 4-manifolds have handlebody decompositions with no 3- or 4-handles.

Proposition. *If D is a ribbon disk in B^4 and K is a piecewise linear knot in $\partial \widetilde{B}^4$, then the Poénaru 4-manifold $P^4(D, K)$ has a handlebody decomposition with no 3- or 4-handles.*

Proof. Let N, A, T and g be as in the paragraph describing the construction of $P^4(D, K)$. To prove the Proposition, we will established two assertions.

(a) $\mathrm{cl}(B^4 - N)$ has a handlebody decomposition with no 3- or 4-handles.

(b) There is a piecewise linear homeomorphism from $\widetilde{B}^4$ to $B^3 \times [0, 1]$ which identifies T with a subset $T_0 \times \{0\}$ of $B^3 \times \{0\}$ so that $B^3 \times [0, 1]$ is obtained from $T_0 \times [0, 1]$ by attaching 1- and 2-handles to $(\partial T_0) \times [0, 1]$.

The proof of the Proposition is then completed by noting that since $\mathrm{cl}(B^4 - N)$ is piecewise linearly homeomorphic to $(T_0 \times [0, 1]) \cup_g \mathrm{cl}(B^4 - N)$, then by assertion (a), $(T_0 \times [0, 1]) \cup_g \mathrm{cl}(B^4 - N)$ has a handlebody decomposition with no 3- or 4-handles. Furthermore, by assertion (b), $(B^3 \times [0, 1]) \cup_g \mathrm{cl}(B^4 - N)$ is obtained from $(T_0 \times [0, 1]) \cup_g \mathrm{cl}(B^4 - N)$ by attaching 1- and 2-handles. We conclude that $(B^3 \times [0, 1]) \cup_g \mathrm{cl}(B^4 - N) = P^4(D, K)$ has a handlebody decomposition with no 3- or 4-handles.

We now demonstrate assertion (a): $\mathrm{cl}(B^4 - N)$ has a handlebody decomposition with no 3- or 4-handles. (Evidently, a related fact is proved in [8], though the language there is quite different.) We can identify B^4 with $B^3 \times [0, 1]$ so that $\partial D \subset \mathrm{int}(B^3) \times \{1\}$. Furthermore, we can assume that the ribbon disk D is positioned in a special way that we now describe. D arises from a map $f: B^2 \to \mathrm{int}(B^3) \times \{1\}$ with singular set equal to the union of a pairwise disjoint collection of arcs $A_1, A_2, \ldots, A_k$ such that for $1 \leqslant i \leqslant k$, $f^{-1}(A_i)$ is the union of two disjoint arcs A_i' and A_i'' in B^2 where $A_i' \subset \mathrm{int}(B^2)$, $A_i'' \cap (\partial B^2) = \partial A_i''$, and f maps each of A_i' and A_i'' homeomorphically onto A_i.

We impose a "collared" handlebody decomposition on B^2 as follows. The 0-handles are disjoint disks $E_1, E_2, \ldots, E_k$ in $\mathrm{int}(B^2)$ such that $A_i' \subset \mathrm{int}(E_i)$ and $E_i \cap A_j'' = \emptyset$ for $1 \leqslant i, j \leqslant k$. For $1 \leqslant i \leqslant k$, we add an exterior collar to E_i to obtain a slightly larger disk E_i^+ in $\mathrm{int}(B^2)$ so that $E_1^+, E_2^+, \ldots, E_k^+$ are pairwise disjoint and are disjoint from $A_1'', A_2'', \ldots, A_k''$. Next we connect the k disks $E_1^+, E_2^+, \ldots, E_k^+$ with $k-1$ disjoint 1-handles or "bands" $F_1, F_2, \ldots, F_{k-1}$ in $\mathrm{int}(B^2) - \bigcup_{i=1}^k \mathrm{int}(E_i^+)$. Set

$$G = \left(\bigcup_{i=1}^{k} E_i^+\right) \cup \left(\bigcup_{j=1}^{k-1} F_j\right).$$

Then G is a disk in $\mathrm{int}(B^2)$. For $1 \leqslant j \leqslant k-1$, each of the sets $(\partial F_j) \cap (\bigcup_{j=1}^k E_i^+)$ and $(\partial F_j) \cap (\partial G)$ is the union of two disjoint arcs in ∂F_j, and these four arcs subdivide ∂F_j and have disjoint interiors. We add an exterior collar to G to obtain a slightly larger disk G^+ in $\mathrm{int}(B^2)$. Of course, $B^2 - \mathrm{int}(G^+)$ is an annulus.

To form the ribbon disk D from the map f, we push f "vertically" down the $[0,1]$-fibers of $B^3 \times [0,1]$ and make some minor "horizontal" adjustments to achieve an embedding with the following properties. (We now identify B^2 with its image D.) The 0-handles $E_1, E_2, \ldots, E_k$ lie in the level $B^3 \times \{1/4\}$. For $1 \leqslant i \leqslant k$, the collar $E_i^+ - \mathrm{int}(E_i)$ lies vertically over ∂E_i in the product $B^3 \times [1/4, 1/2]$ so that ∂E_i^+ lies in the level $B^3 \times \{1/2\}$. The 1-handles $F_1, F_2, \ldots, F_{k-1}$ lie in the level $B^3 \times \{1/2\}$. The collar $G^+ - \mathrm{int}(G)$ lies vertically over ∂G in the product $B^3 \times [1/2, 3/4]$ so that ∂G^+ lies in the level $B^3 \times \{3/4\}$. The annulus $D - \mathrm{int}(G^+)$ lies in the product $B^3 \times [3/4, 1]$ so that each level circle of the annulus lies in a $B^3 \times \{t\}$-level and, of course, ∂D lies in $B^3 \times \{1\}$.

Let $\pi : B^3 \times [0,1] \to B^3$ denote projection. The regular neighborhood N of D can be assumed to have the following form:

$$\begin{aligned} N = {} & \big(N_1 \times [1/4 - \delta, 1/4 + \delta]\big) \cup \big(N_2 \times [1/4 + \delta, 1/2 - \delta]\big) \\ & \cup \big(N_3 \times [1/2 - \delta, 1/2 + \delta]\big) \cup \big(N_4 \times [1/2 + \delta, 3/4]\big) \cup N_5 \end{aligned}$$

where N_1, N_2, N_3 and N_4 are regular neighborhoods of $\pi(D \cap (B^3 \times \{t\}))$ in $\mathrm{int}(B^3)$ for $t = 1/4, 3/8, 1/2$ and $5/8$, respectively. N_5 is a regular neighborhood of the annulus $D - \mathrm{int}(G^+)$ in $B^3 \times [3/4, 1]$, and $0 < \delta < 1/8$.

N_1 is a regular neighborhood of the union of the k disks $\pi(E_i)$, $1 \leqslant i \leqslant k$; and N_2 is a regular neighborhood of the union of the k simple closed curves $\pi(\partial E_i)$, $1 \leqslant i \leqslant k$. Thus, N_1 has k components each of which is a 3-ball containing one of the disks $\pi(E_i)$, and N_2 has k components each of which is a solid torus containing one of the simple closed curves $\pi(\partial E_i)$. Moreover, we can assume that $N_2 \subset N_1$, and that $\mathrm{cl}(N_1 - N_2)$ has k components each of which is a 3-ball that intersects $\mathrm{cl}(B^3 - N_1)$ in a pair of disjoint boundary disks. This allows us to view each component of $\mathrm{cl}(N_1 - N_2)$ as a 3-dimensional 1-handle attached to $\mathrm{cl}(B^3 - N_1)$. Hence, $\mathrm{cl}(B^3 - N_2)$ is obtained by attaching k 3-dimensional 1-handles (the components of $\mathrm{cl}(N_1 - N_2)$) to $\mathrm{cl}(B^3 - N_1)$.

Let X denote the union of the simple closed curves ∂E_i^+, $1 \leqslant i \leqslant k$, and the "bands" F_j, $1 \leqslant j \leqslant k$. N_3 is a regular neighborhood of $\pi(X)$. Hence, we can assume that $N_2 \subset N_3$ and that N_3 is obtained from N_2 by attaching $k-1$ 3-dimensional 1-handles, each 1-handle containing one of the disks $\pi(F_j)$. N_4 is a regular neighborhood of $\pi(\partial G)$, and ∂G is obtained from X by removing from X all of F_j except for the two arcs comprising $F_j \cap (\partial G)$ for $1 \leqslant j \leqslant k-1$. It follows that we can assume that $N_4 \subset N_3$, and that $\mathrm{cl}(N_3 - N_4)$ has $k-1$ components each of which is a 3-ball that intersects $\mathrm{cl}(B^3 - N_3)$ in a boundary annulus. This allows us to view each component of $\mathrm{cl}(N_3 - N_4)$ as a 3-dimensional 2-handle attached to $\mathrm{cl}(B^3 - N_3)$. Hence, $\mathrm{cl}(B^3 - N_4)$ is obtained by attaching $k-1$ 3-dimensional 2-handles (the components of $\mathrm{cl}(N_3 - N_4)$) to $\mathrm{cl}(B^3 - N_3)$.

The following seven assertions clearly imply that $\mathrm{cl}(B^4 - N)$ has a handlebody decomposition involving no 3- or 4-handles.

(i) $Y_0 = B^3 \times [0, 1/4 - \delta]$ is a 4-ball and can, thus, be regarded as a 0-handle.
(ii) $Y_{0+} = Y_0 \cup (\mathrm{cl}(B^3 - N_1) \times [1/4 - \delta, 1/2 - \delta])$ is homeomorphic to Y_0.
(iii) $Y_1 = \mathrm{cl}(B^3 \times [0, 1/2 - \delta] - N)$ is obtained from Y_{0+} by attaching 1-handles.
(iv) $Y_{1+} = Y_1 \cup (\mathrm{cl}(B^3 - N_3) \times [1/2 - \delta, 3/4])$ is homeomorphic to Y_1.
(v) $Y_2 = \mathrm{cl}(B^3 \times [0, 3/4] - N)$ is obtained from Y_{1+} by attaching 2-handles.
(vi) $Y_{2+} = Y_2 \cup (\mathrm{cl}(B^3 - N_4) \times [3/4, 1])$ is homeomorphic to Y_2.
(vii) $\mathrm{cl}(B^4 - N)$ is homeomorphic to Y_{2+}.

Assertions (i), (ii), (iv) and (vi) are immediate.

To prove assertion (iii), observe that $Y_1 = Y_{0+} \cup (\mathrm{cl}(N_1 - N_2) \times [1/4 + \delta, 1/2 - \delta])$. Since $\mathrm{cl}(N_1 - N_2)$ can be viewed as the union of k 3-dimensional 1-handles attached to $\mathrm{cl}(B^3 - N_1)$, then $\mathrm{cl}(N_1 - N_2) \times [1/4 + \delta, 1/2 - \delta]$ can be viewed as the union of k 4-dimensional 1-handles attached to Y_{0+} along $(\partial \mathrm{cl}(B^3 - N_1)) \times [1/4 + \delta, 1/2 - \delta]$. Hence, Y_1 is obtained from Y_{0+} by attaching k 4-dimensional 1-handles.

To prove assertion (v), observe that $Y_2 = Y_{1+} \cup (\mathrm{cl}(N_3 - N_4) \times [1/2 + \delta, 3/4])$. Since $\mathrm{cl}(N_3 - N_4)$ can be viewed as the union of $k-1$ 3-dimensional 2-handles attached to $\mathrm{cl}(B^3 - N_3)$, then $\mathrm{cl}(N_3 - N_4) \times [1/2 + \delta, 3/4]$ can be viewed as the union of $k-1$ 4-dimensional 2-handles attached to Y_{1+} along $(\partial \mathrm{cl}(B^3 - N_3)) \times [1/2 + \delta, 3/4]$. Hence, Y_2 is obtained from Y_{1+} by attaching $k-1$ 4-dimensional 2-handles.

Finally, to prove assertion (vii), we observe that the original map $f : B^2 \to B^3 \times \{1\}$ embeds the annulus $B^2 - \mathrm{int}(G^+)$. Hence, there is a piecewise linear ambient isotopy of $B^3 \times \{1\}$ which "drags" $f(\partial G^+)$ through the level circles of the annulus $f(B^2 - \mathrm{int}(G^+))$. This ambient isotopy can be "spread out" as a level preserving piecewise linear homeomorphism h of $B^3 \times [3/4, 1]$ which restricts to the identity on $B^3 \times \{3/4\}$, which carries the "cylinder" $\pi(\partial G^+) \times [3/4, 1]$ onto the annulus $D - \mathrm{int}(G^+)$, and which carries $N_4 \times [3/4, 1]$ onto N_5. (If $h(N_4 \times [3/4, 1]) \neq N_5$ initially, we correct this by redefining N_5.) We extend h over $B^3 \times [0, 3/4]$ via the identity. Then h carries Y_{2+} onto

$$Y_2 \cup \mathrm{cl}\big(B^3 \times [3/4, 1] - N_5\big) = \mathrm{cl}\big(B^3 \times [0, 1] - N\big) = \mathrm{cl}(B^4 - N).$$

This completes the proof of assertion (a): $\mathrm{cl}(B^4 - N)$ has a handlebody decomposition with no 3- or 4-handles.

It remains to demonstrate assertion (b): there is a piecewise linear homeomorphism from $\widetilde{B}^4$ to $B^3 \times [0, 1]$ which identifies T with a subset $T_0 \times \{0\}$ of $B^3 \times \{0\}$ so that $B^3 \times [0, 1]$ is obtained from $T_0 \times [0, 1]$ by attaching 1- and 2-handles to $(\partial T_0) \times [0, 1]$. Let C^3 be a 3-ball in $\partial \widetilde{B}^4$ such that $T \subset \mathrm{int}(C^3)$. $C^3 - \mathrm{int}(T)$ has a handlebody decomposition based on T; in other words, C^3 can be obtained by attaching 0-, 1-, 2-

and 3-handles to T. The 0-handles of this decomposition can be eliminated by cancelling them with some 1-handles, and the 3-handles can be eliminated by cancelling them with some 2-handles. These cancellations can be performed without moving T, but then C^3 may be forced to move. At the end of the process, T is still a subset of the (possibly repositioned) 3-ball C^3. (T may no longer be interior to C^3.) Now C^3 is obtained by attaching 1- and 2-handles to T. Since C^3 is a piecewise linear 3-ball in $\partial\widetilde{B}^4$, there is a piecewise linear homeomorphism $k: B^3 \times [0,1] \to \widetilde{B}^4$ such that $k(B^3 \times \{0\}) = C^3$. There is a solid torus T_0 in B^3 such that $k(T_0 \times \{0\}) = T$. It follows that B^3 can be obtained from T_0 by adding 3-dimensional 1- and 2-handles. By "crossing" each of these handles with $[0,1]$, we see that $B^3 \times [0,1]$ can be obtained from $T_0 \times [0,1]$ by attaching 4-dimensional 1- and 2-handles to $(\partial T_0) \times [0,1]$. This proves assertion (b). □

Acknowledgment

The authors wish to thank Andrew Casson and Martin Scharlemann for a helpful discussion concerning Conjectures 2, 2A and 2B and the Proposition.

References

[1] F.D. Ancel and C.R. Guilbault, Compact contractible n-manifolds have arc spines ($n \geqslant 5$), Pacific J. Math. 168 (1995) 1–10.
[2] R.J. Daverman, Decompositions of Manifolds (Academic Press, Orlando, FL, 1986).
[3] R.J. Daverman and W.T. Eaton, An equivalence for the embeddings of cells in a 3-manifold, Trans. Amer. Math. Soc. 145 (1969) 369–381.
[4] C.R. Guilbault, Some compact contractible manifolds containing disjoint spines, Topology 34 (1995) 99–108.
[5] C.H. Giffen, Disciplining dunce hats in 4-manifolds, unpublished manuscript, 1976.
[6] B. Mazur, A note on some contractible 4-manifolds, Ann. of Math. 73 (1961) 221–228.
[7] V. Poénaru, Les décompositions de l'hypercube en produit topologique, Bull. Soc. Math. France 88 (1960) 113–129.
[8] T. Yanagawa, On ribbon 2-knots II, Osaka J. Math. 6 (1969) 465–473.