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# The locally flat approximation of cell-like embedding relations

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### Abstract

By means of M.A. Štan'ko's clever unknotting technique we show that every embedding $f: M^{n-1} \to N^n$ $(n \geqq 5)$ from a topological $(n\text{-}1)$-manifold $M^{n-1}$ into a topological $n$-manifold $N^n$ can be approximated by locally flat embeddings. A serious technical difficulty forces us to work in the category of cell-like embedding relations rather than single-valued embeddings. The bonus of this enforced generality is that the results obtained will surely have application to the study of cell-like decompositions and generalized manifolds.

## 1. Introduction

By means of M.A. Štan'ko's unknotting technique [S1], [S2], we show that every embedding $f: M^{n-1} \to N^n$ $(n \geqq 5)$ from a topological $(n\text{-}1)$-manifold $M^{n-1}$ into a topological $n$-manifold $N^n$ can be approximated by locally flat embeddings. Others have reported tentative success in the same venture [Š3], [BES]; however, so far as we can determine, previous proofs have all run afoul of an unexpected pathology in the topology of codimension-one embeddings discovered by R. J. Daverman [D1]. (See the remark in Section 5.3.) We circumvent the difficulty by passing from the category of (single-valued) embeddings into the category of (multiple-valued) cell-like embedding relations (Appendix I) where the flexibility of "large points" allows us to destroy Daverman's pathology. (See the Embedding Theorem of Section 5.)

Štan'ko's original paper [Š1] and Cannon's Park City paper [C1] (where cell-like relations are introduced) are the major sources of our techniques. Although the reader may wish to refer to Štan'ko's paper, we give a complete exposition of his technique here.

As a trade-off for the unfamiliar demands we make of the reader, we shall in the main body of the paper consider only the simplest case, that of

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a cell-like embedding relation $R: S^{n-1} \to S^n (n \geqq 5)$. We have made every effort, however, to state the proofs in such a way that they generalize almost without change to many more situations. We discuss the basic generalizations in Section 6. One corollary is a one-sided 1-LCC taming theorem for $(n - 1)$-manifolds in an $n$-manifold. Seebeck [S] has a proof of a stronger one-sided result. A. V. Černavskiĭ [Č] has also claimed a proof, but it apparently has not appeared in print. Daverman [D2] has used our main result in proving that every crumpled $n$-cell $(n \geqq 5)$ is a closed $n$-cell complement.

## 2. Conventions and terminology

(2.1) All spaces are assumed locally compact and separable metric.

(2.2) We take *Euclidean n-space* $E^n$ to be the space $\{t = (t_1, \cdots, t_n) | t_i \in \mathbf{R}\}$ of $n$-tuples of real numbers, with norm $|t| = \max_i |t_i|$ and metric $d(s, t) = |s - t|$ induced by the norm. We write $S^n$ (*the n-spere*) for the subspace $\{t \in E^{n+1} \mid |t| = 1\}$ of $E^{n+1}$ and $B^n$ (*the n-ball*) for the subspace $\{t \in E^n \mid |t| \leqq 1\} = [-1, 1] \times \cdots \times [-1, 1]$ ($n$ times) of $E^n$. We write $I = [-2, 2]$ and $I^n = 2B^n = [-2, 2] \times \cdots \times [-2, 2]$ ($n$ times). We think of $I^k$ as $I^k \times \{0\} \subset I^k \times I^m = I^{k+m} \subset E^{k+m}$.

(2.3) An *n-manifold* is a space, each point of which has a neighborhood homeomorphic with Euclidean $n$-space $E^n$. Thus our manifolds have no boundary.

(2.4) We use the $(+)$ symbol to denote finite disjoint topological union so that a *disk* $(+)$ is a space having finitely many components each of which is a disk. Similarly, a *pinched-disk* $(+)$ is a space having finitely many components each of which is a pinched disk (defined in Section 4), etc. This $(+)$ convention allows us to describe constructions essentially component by component and to economize with our definitions and notations. For example we generally use the same kind of notation to denote a disk $(+)$ that we use to denote a disk.

## 3. The 1-LCC Approximation Theorem

Cell-like embedding relations are defined at the end of Appendix I. We define a cell-like embedding relation $R: S^{n-1} \to S^n$ to be 1-LCC if for each $x \in S^{n-1}$ and each neighborhood $U$ of $R(x)$ in $S^n$ there is a neighborhood $W$ of $R(x)$ in $S^n$ such that each loop in $W - \operatorname{Im} R$ shrinks to a point in $U - \operatorname{Im} R$. Our main result is the following.

1-LCC Approximation Theorem. *Suppose* $R: S^{n-1} \to S^n$ $(n \geqq 5)$ *is a cell-*

*like embedding relation and* $L$ *is a neighborhood of* $R$ *in* $S^{n-1}\times S^n$. *Then* $L$ *contains a* 1-LCC *cell-like embedding relation* $R'\colon S^{n-1}\to S^n$.

COROLLARY. *The neighborhood* $L$ *contains a locally flat embedding* $r\colon S^{n-1}\to S^n$.

For $n=3$, the corollary has been proved by R. H. Bing [B1]. The theorem and corollary are unresolved for $n=4$.

*Proof of the corollary.* By [C1, Theorem 38] (see [C2] for an alternative and more general approach), the set $(-1,1)\times L\subset(-1,1)\times S^{n-1}\times S^n$ contains a cell-like embedding relation $R''\colon(-1,1)\times S^{n-1}\to S^n$ which extends the 1-LCC embedding relation $R'=R''|(\{0\}\times S^{n-1})$. By [Si] (see [C1, Theorem 55]) we may assume $R''$ is a function except on $\{0\}\times S^{n-1}$. Any of the (bicollared) embeddings $r=R''_t$, $t\neq 0$, satisfies the conclusion of the corollary.

The 1-LCC Approximation Theorem will be proved by means of the Basic Lemma stated below. The proof of the Basic Lemma will occupy the main body of the paper. First we need a definition.

*Definition.* Suppose $R\colon S^{n-1}\to S^n$ is a cell-like embedding relation. Then we denote by $\pi_R\colon S^n\to S^n/R$ the identification projection with nondegenerate point preimages equal to the nondegenerate point image of $R$. Let $f\colon S^1\to S^n-\operatorname{Im}R$ be a loop. Let $f^*\colon B^2\to S^n/R$ be a continuous function extending $\pi_R\cdot f$ such that $\operatorname{Im}f^*$ misses one of the two complementary domains of $\operatorname{Im}(\pi_R\cdot R)$ in $S^n/R$. (That $S^n/R-\operatorname{Im}(\pi_R\cdot R)$ has two components is a direct consequence of [C1, Theorems 27 and 29].) Then $F=\pi_R^{-1}\circ f^*\colon B^2\to S^n$ is called an *R-disk bounded by f*. In the proof of the 1-LCC Approximation Theorem below we shall prove that every loop $f\colon S^1\to S^n-\operatorname{Im}R$ near a point image of $R$ bounds a "small" $R$-disk $F$: we define the $R$-diameter of a set $X\subset S^n$ by the equation $R\text{-diam}(X)=\inf\{\varepsilon>0\mid \text{for some } s\in S^{n-1},\ X\subset\varepsilon\circ R\circ\varepsilon(s)\}$; then we define $R\text{-diam}(F)=R\text{-diam}(\operatorname{Im}F)$.

BASIC LEMMA. *Suppose* $R\colon S^{n-1}\to S^n$ $(n\geqq 5)$ *is a cell-like embedding relation,* $F\colon B^2\to S^n$ *is an R-disk, and* $L$ *and* $M$ *are neighborhoods of* $R$ *and* $F$, *respectively. Then* $L$ *and* $M$ *contain a cell-like embedding relation* $R'''\colon S^{n-1}\to S^n$ *and a continuous function* $f'\colon B^2\to S^n$ *with disjoint images.*

*Proof of the* 1-LCC *Approximation Theorem.* Let $f_1,f_2,\cdots\colon S^1\to S^n$ denote a countable set of simple closed curves dense in the space of loops in $S^n$.

Let $R_0=R$ and let $L_0\subset L$ be a compact neighborhood of $R_0$. Assume inductively that cell-like embedding relations $R_0,\cdots,R_{i-1}\colon S^{n-1}\to S^n$, compact neighborhoods $L_0\supset R_0,\ \cdots,\ L_{i-1}\supset R_{i-1}$ in $S^{n-1}\times S^n$ and continuous functions $g_1,\ \cdots,\ g_{i-1}\colon B^2\to S^n$ bounded by $f_1,\ \cdots,\ f_{i-1}$ have been chosen satisfying the following four conditions for $j = 0, \cdots, i-1$:

($1_j$) $R_j \subset \operatorname{Int} L_j \subset L_j \subset (1/j) \circ R_j \circ (1/j)$;

($2_j$) $L_j$ is *slice trivial* in $L_{j-1}$ (i.e., if $x \in S^{n-1}$, then $L_j(x)$ contracts in $(\operatorname{Int} L_{j-1})(x)$; see [C1, Lemma 16]);

($3_j$) $L_j^{-1} \circ L_j \subset (1/j)$; and

($4_j$) If $f_j$ bounds an $R_{j-1}$-disk and if

$$\varepsilon_j = \inf \{R_{j-1}\text{-diam}(F) \mid F \text{ is an } R_{j-1}\text{-disk bounded by } f_j\},$$

then $R_{j-1}$-diam(Im $g_j$) $< 2\varepsilon_j$ and $\operatorname{Im} L_j \cap \operatorname{Im} g_j = \varnothing$.

Choose $R_i$, $L_i$, and $g_i$ as follows. If $f_i$ bounds an $R_{i-1}$-disk and $\varepsilon_i$ is defined as above, it is an easy consequence of the Basic Lemma that there exists a cell-like embedding relation $(R_i: S^{n-1} \to S^n) \subset \operatorname{Int} L_{i-1}$ and a continuous function $g_i: B^2 \to S^n$ bounded by $f_i$ such that $\operatorname{Im} R_i \cap \operatorname{Im} g_i = \varnothing$ while $R_{i-1}$-diam(Im $g_i$)$<2\varepsilon_i$. There is a compact neighborhood $L_i$ of $R_i$ in $(\operatorname{Int} L_{i-1}) \cap [(1/i) \circ R_i \circ (1/i)]$ by result (5) of Appendix I on continuous relations; thus ($1_i$) may be satisfied. Condition ($2_i$) may be satisfied by [C1, Lemma 16]. Since $R_i^{-1} \circ R_i = \mathrm{id} \subset (1/i)$, condition ($3_i$) may be satisfied by the the Composition Theorem (result (6) of Appendix I). Since $\operatorname{Im} R_i \cap \operatorname{Im} g_i = \varnothing$, condition ($4_i$) may be satisfied.

If $f_i$ bounds no $R_{i-1}$-disk, condition ($4_i$) becomes vacuous; one may take $R_i = R_{i-1}$, $g_i$ arbitrary, and $L_i$ satisfying ($1_i$), ($2_i$), and ($3_i$) as above. This completes the inductive step.

Define $(R' = \bigcap_{i=0}^{\infty} L_i): S^{n-1} \to S^n$. We claim that $R'$ is a 1-LCC cell-like embedding relation (clearly contained in $L$).

(i) $R'$ is a proper relation; indeed $R'$ is an intersection of compact sets, hence compact, hence proper by result (5) of the appendix on continuous relations.

(ii) $R'$ is cell-like; indeed $R'(x)$ $(x \in S^{n-1})$ has the basic neighborhood system $L_0(x) \supset L_1(x) \supset \cdots$ with $L_i(x)$ compact, nonempty, and contractible in $L_{i-1}(x)$ by condition ($2_i$); thus $R'(x)$ is cell-like.

(iii) $R'$ is injective; indeed,

$$(R')^{-1} \circ R' \subset \textstyle\bigcap_i L_i^{-1} \circ L_i \subset \bigcap_i (1/i) = \mathrm{id}_{S^{n-1}};$$

hence point images of $R'$ are disjoint.

(Conditions (i), (ii), and (iii) show $R'$ is a cell-like embedding relation.)

(iv) The following argument shows that $R'$ is 1-LCC:

Suppose a point $x \in S^{n-1}$ and a neighborhood $U$ of $R'(x)$ in $S^n$ given. Our task is to find a neighborhood $W$ of $R'(x)$ in $S^n$ such that each loop in $W - \operatorname{Im} R'$ contracts in $U - \operatorname{Im} R'$.

We first use the Composition Theorem to supply some estimates. Since $U \supset R'(x) = (\mathrm{id} \circ R' \circ \mathrm{id}) \circ (\mathrm{id} \circ R'^{-1} \circ \mathrm{id}) \circ (\mathrm{id} \circ R' \circ \mathrm{id})(x)$, it follows from the Composition Theorem that there is a positive number $\alpha$ and a positive integer $I$ satisfying

$$(1) \qquad U \supset (2\alpha \circ L_I \circ 2\alpha) \circ (2\alpha \circ L_I^{-1} \circ 2\alpha) \circ (\alpha \circ R' \circ \alpha)\,(x)\,.$$

Set $\beta = \alpha/2$ and choose a positive integer $J > 2/\alpha$. Then $i > J$ implies

$$\begin{aligned}(2) \qquad \beta \circ R' \circ \beta(x) &\subset (\alpha/2) \circ L_{i-1} \circ (\alpha/2)(x) \\ &\subset (\alpha/2) \circ (\alpha/2 \circ R_{i-1} \circ \alpha/2) \circ (\alpha/2)(x) \text{ by } (1_{i-1}) \\ &= \alpha \circ R_{i-1} \circ \alpha(x)\,.\end{aligned}$$

Having chosen $\alpha$, $\beta$, $I$, and $J$, we may take an $(n-1)$-cell neighborhood $D$ of $x$ in $\beta(x)$, a compact neighborhood $V$ of $R'(x)$ in $\beta \circ R' \circ \beta(x)$ intersecting $\mathrm{Im}\, R'$ only in $R'(\mathrm{Int}\, D)$, and a compact neighborhood $W$ of $R'(x)$ which contracts in $V$ (recall that $R'(x)$ is cell-like). Then we shall show that each loop $f\colon S^1 \to S^n$ in $W - \mathrm{Im}\, R'$ contracts in $U - \mathrm{Im}\, R'$.

Pick $K > \mathrm{Max}\{I, J\}$ so large that if $i > K$, then

$$(3) \qquad R_{i-1}(D) \subset \beta R' \beta(x)\,,$$

$$(4) \qquad R_{i-1}(S^{n-1} - \mathrm{Int}\, D) \cap V = \varnothing\,, \quad \text{and}$$

$$(5) \qquad \mathrm{Im}\, f \cap \mathrm{Im}\, R_{i-1} = \varnothing\,.$$

Pick $i > K$ such that the loop $f_i$ is homotopic to $f$ in $W - \mathrm{Im}\,(R' \cup R_{i-1})$ (recall (5)). We shall show that $g_i\colon B^2 \to S^n$ has image in $U - \mathrm{Im}\, R'$. This will complete the proof.

Since $W$ contracts in $V$, there is a continuous extension $g\colon B^2 \to V$ of $f_i$. Let $\pi = \pi_{R_{i-1}}\colon S^n \to S^n/R_{i-1}$ and identify $S^{n-1}$ with $\mathrm{Im}\,(\pi \circ R_{i-1})$ via the embedding $\pi \circ R_{i-1}$. The set $\pi \circ f_i(S^1)$ lies in one of the two components of $(S^n/R_{i-1}) - S^{n-1}$, and the set $\pi \circ g(B^2)$ intersects $S^{n-1}$ only in the $(n-1)$-cell $\pi \circ R_{i-1}(D)$ (by (4)). By the Tietze Extension Theorem, there is a continuous function $f^*\colon B^2 \to S^n/R_{i-1}$ extending $\pi \circ f_i$ whose image lies in $\pi \circ g(B^2) \cup \pi \circ R_{i-1}(D)$ and misses one component of $(S^n/R_{i-1}) - S^{n-1}$. Set $F = \pi^{-1} \circ f^*$.

The relation $F\colon B^2 \to S^n$ is an $R_{i-1}$-disk bounded by $f_i$. Since $\mathrm{Im}\, f^* \subset \pi \circ g(B^2) \cup \pi \circ R_{i-1}(D)$, it follows that $\mathrm{Im}\, F \subset g(B^2) \cup R_{i-1}(D) \subset \beta \circ R' \circ \beta(x)$ (by (3) and the choice of $V \supset g(B^2)$). But $\beta \circ R' \circ \beta(x) \subset \alpha \circ R_{i-1} \circ \alpha(x)$ (by (2)). Hence $g_i$ is a singular disk in $S^n - L_i$ bounded by $f_i$ and lying, for some $y \in S^{n-1}$, in the set $2\alpha \circ R_{i-1} \circ 2\alpha(y)$ (by $(4_i)$). It remains only to show that $2\alpha \circ R_{i-1} \circ 2\alpha(y) \subset U$.

Since $\varnothing \neq \mathrm{Im}\, f_i \subset \alpha \circ R' \circ \alpha(x) \cap 2\alpha \circ R_{i-1} \circ 2\alpha(y)$, it follows that

$$2\alpha \circ R_{i-1} \circ 2\alpha(y) \subset (2\alpha \circ R_{i-1} \circ 2\alpha)[(2\alpha \circ R_{i-1}^{-1} \circ 2\alpha) \circ (\alpha \circ R' \circ \alpha)(x)]\,.$$

But this latter set lies in $U$ (by (1)).

Our proof that $R'$ is 1-LCC is complete.

## 4. Štan'ko's unknotting technique

We shall eventually prove the Basic Lemma by Štan'ko's unknotting technique. Linked 1-handles in Euclidean 3-space $E^3$ can be unlinked simply by pulling one through the other. Štan'ko's basic move is the product extension to high dimensions of this simple unlinking procedure. In high dimensions, however, the basic move may introduce singularities in the space being unknotted. Štan'ko's beautiful accomplishment was the realization that the singularities can often be removed by a meshed sequence of basic moves, each succeeding move removing the singularities introduced by its predecessor, all singularities disappearing in the limit.

In this section we first describe the basic Štan'ko move abstractly without direct reference to the Basic Lemma in order to highlight the simplicity of the technique and to clarify the manner in which singularities arise in high dimensions. We then describe the structures (pinched disks, Štan'ko complexes) which guide Štan'ko moves. Finally, in Section 5 we put everything together in a proof of the Basic Lemma.

4.1. *The basic Štan'ko move.* The basic move is described in terms of a special homeomorphism of the $n$-cube $I^n$:

Certain subsets of $I^2$ are of particular importance to us (Figure 1).

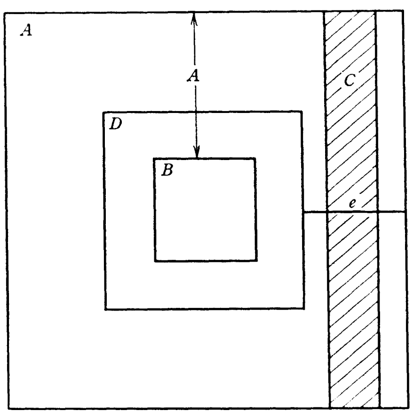


FIGURE 1

$$A = [-2, 2] \times [-2, 2] - \left(-\frac{1}{2}, \frac{1}{2}\right) \times \left(-\frac{1}{2}, \frac{1}{2}\right) \subset I^2 ;$$

$$B = \left[-\frac{1}{2}, \frac{1}{2}\right] \times \left[-\frac{1}{2}, \frac{1}{2}\right] \subset I^2 ;$$

$C = [5/4, 7/4] \times [-2, 2] \subset I^2$ ;
$D = [-1, 1] \times [-1, 1] \subset I^2$ ;
$e = \left[\frac{1}{2} \times \{0\}\right] \subset I^2$ .

Now let $n \geqq 3$ and consider the sets $\mathcal{A} = A \times I^{n-2}$, $\mathcal{B} = B \times I^{n-2}$, $\mathcal{C} = C \times [-1, 1] \times I^{n-3}$, $\mathcal{D} = D \times \{0\} \subset I^n$, and $e \times I = e \times I \times \{0\} \subset I^n$. These five sets will play special roles in our constructions. In the case $n = 3$, $\mathcal{B}$ and $\mathcal{C}$ may be viewed as disjoint 1-handles cutting through $I^3$ perpendicular to each other. The basic Stan'ko move restricted to $I^3$ will simply pull $C \times [-1, 1] = \mathcal{C} \cap I^3$ through $B \times I = \mathcal{B} \cap I^3$.

We now describe the special homeomorphism ($\Phi = \Phi_n$): $I^n \to I^n$, fixed on Bd $I^n$, alluded to above; it shifts the set $\mathcal{C}$ relative to the set $\mathcal{B}$ in the following manner. Let $\Phi$: $I \times [0, 1] \to I$ be the isotopy of $I$ which, for fixed $t \in [0, 1]$, fixes $-2$ and 2, shifts the segment $[5/4, 7/4]$ $3t$ units to the left, and is linear on the segments $[-2, 5/4]$ and $[7/4, 2]$. Then, for $s \in I$, $t \in I^{n-1}$, $(s, t) \in I^n$, we define

$$\Phi_n(s, t) = \begin{cases} (\phi(s, 1), t) & \text{if} \quad |t| \in [0, 1] \\ (\phi(s, 2 - |t|), t) & \text{if} \quad |t| \in [1, 2] \, . \end{cases}$$

The case $n = 2$ is pictured in Figure 2. We note the following facts.

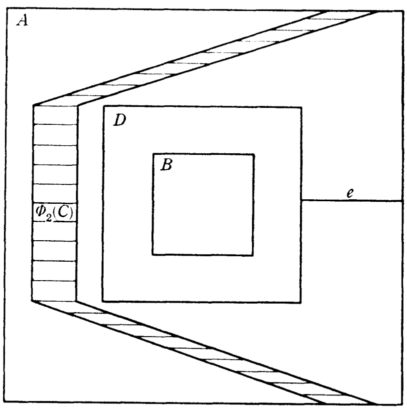


FIGURE 2

(4.1.1) $\Phi_n(C \times [-1, 1] \times [-1, 1]^{n-3}) \subset \Phi_2(C) \times [-1, 1] \times I^{n-3} \subset I^n - \mathcal{B}$ if $n \geqq 2$.

(4.1.2) $\Phi_n(\mathcal{C}) \cap \mathcal{B} \neq \emptyset$ if $n > 3$ .

We describe the *basic move* in terms of $\Phi_n$ as follows. Suppose that $Y$ is an $n$-manifold, $X$ is a closed subset of $Y$, $f$: $I^{n-3} \to Y$ is an embedding, and $f^{-1}(X) \subset \mathcal{B} \cup \mathcal{C}$. Then the continuous function $f^*$: $X \to Y$ defined by

$$f^*(x) = \begin{cases} f \circ \Phi_n \circ f^{-1}(x) & \text{if} \quad x \in f(\mathcal{C}) \\ x & \text{otherwise} \end{cases}$$

is a basic Stan'ko move. If $n \leqq 3$, then $f^*$ is a re-embedding of $X$ in $Y$ because of (4.1.1). For $n > 3$, the map $f^*$ will, in general, not be injective because of (4.1.2). Nevertheless, in both cases the move may undo some self-linking in $X$.

Summarize the above information in a symbol $(A, B, C, D, e)$ called a *template* and generalize all of the above via the $(+)$ convention of (2.4) to obtain templates $(+)$ and homeomorphisms $\Phi = \Phi_n: \mathcal{A} \cup \mathcal{B} \to \mathcal{A} \cup \mathcal{B}$ acting on each component of $\mathcal{A} \cup \mathcal{B}$ just as described above.

We now turn to the structures that will guide basic moves in a manifold $Y$.

4.2. *Pinched disks and Štan'ko complexes.* Let $D$ denote an oriented PL disk; let $L_1, \cdots, L_k, R_1, \cdots, R_k (k \geqq 0)$ denote compatibly oriented disjoint PL subdisks of $\operatorname{Int} D$ irreducibly joined in pairs by disjoint PL arcs $J_1, \cdots, J_k$ in $\operatorname{Int} D$, $\operatorname{Bd} J_j \subset L_j \cup R_j$; let $f: L_1 \cup \cdots \cup L_k \to R_1 \cup \cdots \cup R_k$ denote an orientation reversing homeomorphism which takes $L_j \cap J_j$ to $R_j \cap J_j$. The identification space $D^* = D/f$ is called a *pinched disk.* Set $E = L_1 \cup \cdots \cup L_k \cup R_1 \cup \cdots \cup R_k$ and $J = J_1 \cup \cdots \cup J_k$. Compile the above information into a delta symbol $\Delta = (D, E, J)$ with identification map $(*): D \to D^*$ understood (Figure 3).

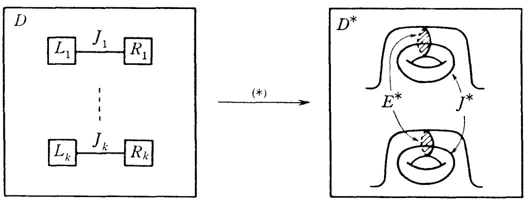


FIGURE 3

Generalize via the $(+)$ convention to obtain *pinched disks* $(+)$ $D^*$ and *delta symbols* $(+)$ $\Delta = (D, E, J)$ (projection $(*): D \to D^*$ understood).

A *branching system* is a system $(\Delta, g): \Delta_0 \xrightarrow{g_0} \Delta_1 \xrightarrow{g_1} \Delta_2 \to \cdots$ where each $\Delta_i$ is a delta symbol $(+)$ $\Delta_i = (D_i, E_i, J_i)$ and each $g_i$ is a PL homeomorphism $g_i: J_i^* \to \operatorname{Bd} D_{i+1}^*$. The identification space $C(\Delta, g) = D_0^* \cup_{g_0} D_1^* \cup_{g_1} D_2^* \cup \cdots$ is called a *Stan'ko complex.* We generally identify each $D_i^*$ with its image in $C(\Delta, g)$. Then we may take $J_i^* = \operatorname{Bd} D_{i+1}^*$ and suppress the maps $g_i$. Thus we write $\Delta = (\Delta, g)$ and $C(\Delta) = C(\Delta, g)$. We combine the identifications $(*): D_i \to D_i^*$ into a single map

$$(*): D_0 \cup D_1 \cup \cdots \text{ (disjoint union)} \longrightarrow D_0^* \cup D_1^* \cup \cdots \; (= C(\Delta)) .$$

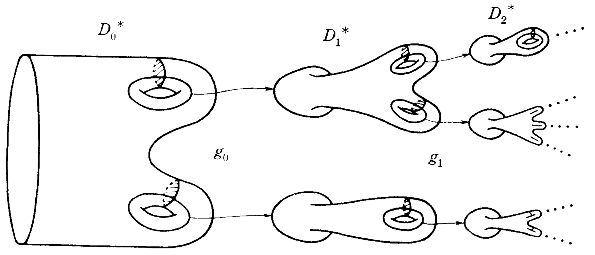


FIGURE 4. Branching system.

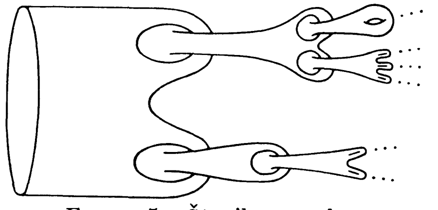

FIGURE 5. Štan'ko complex.

## 5. Proof of the Basic Lemma

5.1. *Setting.* We assume throughout Section 5 the hypotheses of the Basic Lemma and related notations as follows (Figure 6):

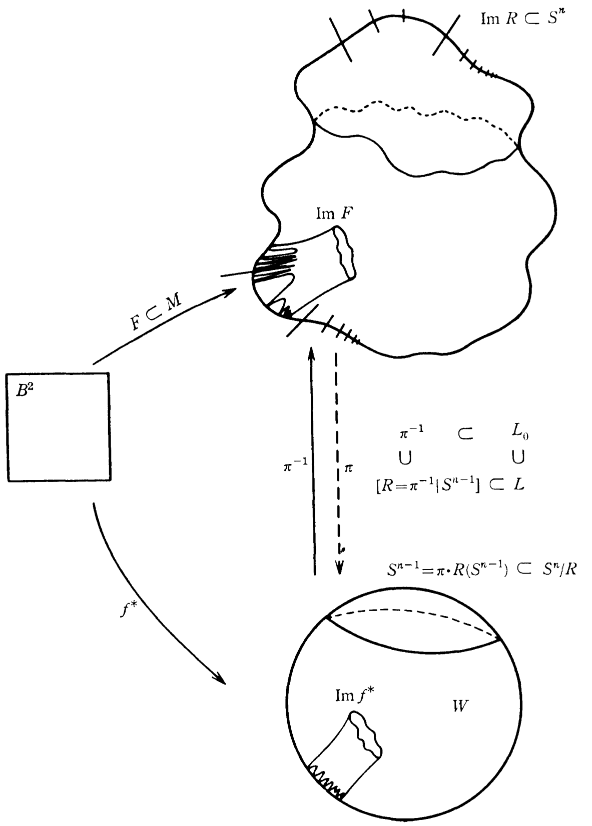


FIGURE 6. Setting for the Basic Lemma.

$R: S^{n-1} \to S^n$, a cell-like embedding relation with associated projection map $\pi = \pi_R: S^n \to S^n/R$;

$(F = \pi^{-1} \circ f^*): B^2 \to S^n$, an $R$-disk with $f^*: B^2 \to S^n/R$ a continuous function and with $F \mid \text{Bd}\, B^2$ a PL embedding;

$L$, a neighborhood of $R$ in $S^{n-1} \times S^n$;

$M$, a neighborhood of $F$ in $B^2 \times S^n$.

We identify $S^{n-1}$ with $\text{Im}\,(\pi \circ R)$ via the homeomorphism $\pi \circ R: S^{n-1} \to \text{Im}\,(\pi \circ R)$ so that $R = \pi^{-1} \mid S^{n-1}$ and fix the notation

$W$, the component of $(S^n/R) - S^{n-1}$ containing $f^*(S^1)$;

$L_0$, a neighborhood of $\pi^{-1}$ in $(S^n/R) \times S^n$ whose restriction to $S^{n-1}$ is $L$.

Looking ahead to calculations which will be made at the end of the proof, we take care of certain essential estimates at once by the Composition Theorem: since

$$(\text{id}_{S^n} \circ \text{id}_{S^n} \circ \pi^{-1} \circ \text{id}_{S^n/R} \circ \pi \circ \text{id}_{S^n} \circ \text{id}_{S^n}) \circ \text{id}_{S^n} \circ \pi^{-1} = \pi^{-1} \subset L_0$$

and $\text{id}_{S^n} \circ \pi^{-1} \circ \text{id}_{S^n/R} \circ f^* = F \subset M$, there is an $\varepsilon > 0$ such that

$$(1) \qquad (\varepsilon \circ \varepsilon \circ \pi^{-1} \circ 2\varepsilon \circ \pi \circ \varepsilon \circ \varepsilon) \circ \varepsilon \circ \pi^{-1} \subset L_0$$

and

$$(2) \qquad \varepsilon \circ \pi^{-1} \circ \varepsilon \circ f^* \subset M\,.$$

We prove the Basic Lemma in three major steps outlined as follows.

Step 1. We associate with the above setting a Štan'ko complex $C(\Delta)$ and a special continuous function $h: C(\Delta) \to S^n/R$ covered by a PL injective function $h': C(\Delta) \to S^n$ (Mapping Theorem and its Addendum).

Step 2. We adjust $R$ so that the map $h'$ of Step 1 may be replaced by a PL embedding $h'': C(\Delta) \to S^n$ (Embedding Theorem).

Step 3. We use the embedded Štan'ko complex $h''\, C(\Delta)$ to guide an infinite Štan'ko move which proves the Basic Lemma.

5.2. MAPPING THEOREM. *There exist a branching system* $\Delta: \Delta_0 \to \Delta_1 \to \Delta_2 \to \cdots$, *with* $D_0$ *of* $\Delta_0 = (D_0, E_0, J_0)$ *equal to* $B^2$, *and a continuous function* $h: C(\Delta) \to S^n/R$ *satisfying the follows conditions*:

$$(1) \qquad h[C(\Delta) - \text{Int}(E_0^* \cup E_1^* \cup \cdots)] \subset W\,.$$

$$(2) \qquad h[D_i^* \cup D_{i+1}^* \cup D_{i+2}^* \cup \cdots] \subset (\varepsilon/i)(S^{n-1})\,.$$

$$(3) \qquad d(h \circ (*) \mid D_0, f^*) < \varepsilon\,.$$

$$(4) \qquad h \circ (*) \mid \text{Bd}\, D_0 = f^* \mid (\text{Bd}\, D_0 = \text{Bd}\, B^2)\,.$$

$$(5) \qquad \text{Diam}\, h(P_i) < \varepsilon/i \textit{ for each component } P_i \textit{ of } D_i^* \cup E_{i-1}^*\,.$$

ADDENDUM. *The map* $h: C(\Delta) \to S^n/R$ *may be chosen so that there is a*

PL *injective map* $h'\colon C(\Delta) \to S^n$ *with* $\pi \circ h' = h$.

The Mapping Theorem is the iterative consequence of the following simple lemma.

LEMMA. *Suppose $D$ is a disk, $f\colon (D, \mathrm{Bd}\, D) \to (\mathrm{Cl}\, W, W)$ is a map of pairs, and $\delta$ is a positive number. Then there exist a delta symbol $\Delta = (D, E, J)$ and a continuous function $h\colon D^* \to \mathrm{Cl}\, W$ satisfying the following conditions*:

$$h[D^* - \mathrm{Int}\, E^*] \subset W . \tag{1'}$$

$$h(E^*) \subset \delta(S^{n-1}) . \tag{2'}$$

$$d(h\circ(^*)|D\ ,\quad f|D) < \delta . \tag{3'}$$

$$h \circ (^*)|\mathrm{Bd}\, D = f|\mathrm{Bd}\, D . \tag{4'}$$

$$\mathrm{Diam}\, h(P) < \delta \quad \textit{for each component } P \textit{ of } E^* \textit{ or } J^* . \tag{5'}$$

*Proof of lemma.* The set $W$ is $0 -$ lc and $1 -$ lc [C1, Theorem 34]; the set $\mathrm{Cl}\, W = S \cup W$ is 1-ULC (see the proof that $R'$ is 1-LCC in the proof of the 1-LCC Approximation Theorem, Section 3). Hence we may choose positive numbers $\alpha < \beta < \gamma < \delta/4$ such that

(i) $\gamma$ loops in Cl $W$ bound singular $\delta/4$-disks in Cl $W$ (Cl $W$ is 1-ULC);

(ii) $\beta$ loops in $W$ bound orientable, singular $\gamma$-disks-with-handles in $W$ ($W$ is 1-ulc);

(iii) two points within $\alpha$ of one another in $W$ are joined by $\beta/2$-arcs in $W$ ($W$ is 0-ulc).

Triangulate $D$ with mesh so small that the image under $f$ of each simplex has diameter less than $\alpha$. Let $D^{(0)}$, $D^{(1)}$, and $D^{(2)}$ denote the skeletons of the triangulation.

Since $\mathrm{Im}\, f \subset \mathrm{Cl}\, W$, we may define $h\,|\,|D^{(0)}|\colon |D^{(0)}| \to W$ equal to $f$ on $|D^0| \cap \mathrm{Bd}\, D$ and so near $f\,|\,|D^0|$ that vertices of the same simplex have images in $W$ within $\alpha$ of one another and within $\alpha$ of their image under $f$.

By (iii), we may extend $h\,|\,|D^0|$ to $h\,|\,|D^{(1)}|\colon |D^{(1)}| \to W$ equal to $f$ on Bd $D$ in such a manner that the image of each 1-simplex has diameter less than $\beta/2$.

By (ii), for each $\sigma \in D^{(2)}$ there exist an orientable disk-with-handles $H_\sigma$ bounded by Bd $\sigma$ and a continuous extension $h|H_\sigma\colon H_\sigma \to W$ of $h|\mathrm{Bd}\,\sigma$ mapping $H_\sigma$ to a set of diameter less than $\gamma$.

In the interior of each $H_\sigma$ identify complete sets $J_\sigma$ and $K_\sigma$ of handle curves: $J_\sigma$ (resp., $K_\sigma$) is a finite disjoint union of simple closed curves, each meeting $K_\sigma$ (resp., $J_\sigma$) transversely in a single point, $J_\sigma$ and $K_\sigma$ maximal.

By (i), there exist for each $\sigma \in D^{(2)}$ a finite disjoint union $E_\sigma$ of disks having boundary $K_\sigma$ and a continuous extension $h|E_\sigma: E_\sigma \to \mathrm{Cl}\, W$ of $h|K_\sigma$ taking each component of $E_\sigma$ to a set of diameter less than $\delta$.

Define

$$D^* = |D^1| \cup \bigcup_\sigma (H_\sigma \cup E_\sigma)\,, \quad E^* = \bigcup_\sigma E_\sigma\,, \quad J^* = \bigcup_\sigma J_\sigma\,.$$

Then $D^*$ is the pinched disk of a delta symbol $\Delta = (D, E, J)$, $(*)\big|\,|D^{(1)}| = \mathrm{id}$, $(*)$: $\sigma \to H_\sigma \cup E_\sigma$. Then $h$ clearly satisfies (1′), (3′), (4′), and (5′). If (2′) is not already satisfied, it can only be because some component of $E^* \cup J^*$ has image missing $S^{n-1} = \mathrm{Bd}\, W$. The preimage of this component in $E \cup J$ may simply be deleted from $E$ and $J$.

*Proof of the Mapping Theorem.* Choose a sequence $\delta_0 > \delta_1 > \delta_2 > \cdots$ of positive numbers such that

(i) $4\delta_i < \varepsilon/i$ for $i \geqq 0$.

(ii) $\delta_{i+1}$ loops in Cl $W$ bound singular $\delta_i$ disks in Cl $W$ (Cl $W$ is 1-ULC) for $i \geqq 0$.

By the lemma, there exist a delta symbol $\Delta_0 = (D_0, E_0, J_0)$, $D_0 = B^2$, and a continuous function $h|D_0^*: D_0^* \to \mathrm{Cl}\, W$ satisfying, for $j = 0$ and $f_j = f|D_0$, the conditions

($1_j$) $h[D_j^* - \mathrm{Int}\, E_j^*] \subset W$,

($2_j$) $h(E_j^*) \subset \delta_{j+1}(S^{n-1})$,

($3_j$) $d[h \circ (*)\,|\,D_j, f_j] < \delta_{j+1}$,

($4_j$) $h \circ (*)|\,\mathrm{Bd}\, D_j = f_j\,|\,\mathrm{Bd}\, D_j$, and

($5_j$) $\mathrm{Diam}\, h(P) < \delta_{j+2}$ for each component $P$ of $E_j^*$ or $J_j^*$.

Assume inductively that $\Delta_0 \to \cdots \to \Delta_{i-1}$, $f_j: D_j \to \mathrm{Cl}\, W$, and $h|D_0^* \cup \cdots \cup D_{i-1}^*$ have been chosen satisfying ($1_j$)-($5_j$) for each $j \in \{0, \cdots, i-1\}$.

For each component $J$ of $J_{i-1}^*$, let $D(J)$ be a disk with boundary $J$. By ($5_{i-1}$), $\mathrm{Diam}\, hJ < \delta_{i+1}$. By (ii), there is a continuous extension $f_i\,|\,D(J): D(J) \to \mathrm{Cl}\, W$ having image of diameter less than $\delta_i$. Let $D_i = \bigcup_J D(J)$ and $[f_i = \bigcup_J f_i\,|\,D(J)]$: $D_i \to \mathrm{Cl}\, W$. By the lemma, there exist a delta symbol (+) $\Delta_i = (D_i, E_i, J_i)$ and a continuous function $h|D_i^*: D_i^* \to \mathrm{Cl}\, W$ satisfying ($1_i$)-($5_i$). This completes the inductive construction of

$$\Delta: \Delta_0 \longrightarrow \Delta_1 \longrightarrow \Delta_2 \longrightarrow \cdots \quad \text{and} \quad h: C(\Delta) \longrightarrow S^n/R\,.$$

Conditions (1), (3), and (4) of the Mapping Theorem are obviously satisfied. For each component $P \cup Q$ of $D_i^* \cup E_{i-1}^*$ $(i \geqq 1)$, $P \subset D_i^*$, $Q \subset E_{i-1}^*$, we have

$$\mathrm{Diam}\, hP \leqq 2d\big(h \circ (*)|P, f_i|P\big) + \mathrm{Diam}\, f_i(P) < 3\delta_i \quad \text{and } \mathrm{Diam}\, hQ < \delta_{i+1}\,;$$

thus (5) is satisfied. Also, for $i \geqq 1$,

$$d(h(P), S^{n-1}) \leqq d(h(P), h(E^*_{i-1})) + \delta_i \text{ (by } (2_{i-1})) = \delta_i$$

since $h(P) \cap h(E^*_{i-1}) \neq \emptyset$. Thus $h(P) \subset 4\delta_i(S^{n-1}) \subset (\varepsilon/i)(S^{n-1})$ and (2) is satisfied.

*Proof of the addendum.* Suppose we have $\Delta$ and $h$ satisfying the five conclusions of the Mapping Theorem with $\pi^{-1} \circ h | \operatorname{Bd} B^2 = F | \operatorname{Bd} B^2$ a PL embedding. We now show how to adjust $h$ so as to satisfy the Addendum.

There is a closed neighborhood $N$ of $h$ in $C(\Delta) \times S^n/R$ such that if $h''\colon C(\Delta) \to S^n/R$ lies in $N$, is continuous, and equals $h$ on $\operatorname{Bd} D_0$, then $h''$ satisfies the same five conclusions.

Since $\pi \circ (\pi^{-1} \circ h) = h \subset N$, it follows from the corollary to the Composition Theorem (result (7) of the appendix on continuous relations) that there is a neighborhood $N'$ of $\pi^{-1} \circ h$ in $C(\Delta) \times S^n$ such that $\pi \circ N' \subset N$. Since $\pi^{-1} \circ h$ is cell-like, the Continuous Approximation Theorem (result (8) of the same appendix) implies the existence of a continuous function $(h'\colon C(\Delta) \to S^n) \subset N'$ with $\pi \circ h' | \operatorname{Bd} B^2 = h | \operatorname{Bd} B^2$. We may adjust $h'$ in $N'$ so as to be PL and in general position. Since $n \geqq 5$, $h'$ is injective. We replace $h$ by $\pi \circ h'$.

5.3. Embedding Theorem. *Let $\Delta$, $h$, and $h'$ be as in the conclusion of the Mapping Theorem and its addendum. Then the $\varepsilon$ neighborhood of* $\mathrm{id}_{S^n}$ *contains a cell-like embedding relation $R'\colon S^n \to S^n$ such that there is a* PL *embedding $h''\colon C(\Delta) \to S^n$ with $R'^{-1} \circ h'' = h'$. (Thus $\pi \circ R'^{-1} \circ h'' = h$.)*

*Remark.* R. J. Daverman [D1, Example 13.3] has described crumpled $n$-cells $C$ in $S^n$ (all $n \geqq 4$) and disjoint PL simple closed curves $J_1$ and $J_2$ in $\operatorname{Int} C$ such that if $D$ is a singular disk in $C$ bounded by $J_1$ and $E$ is a singular disk in $S^n$ bounded by $J_2$, then $D \cap E \neq \emptyset$. Let $F | \operatorname{Bd} B^2\colon \operatorname{Bd} B^2 \to J_1$, and let $R\colon S^{n-1} \to S^n$ take $S^{n-1}$ homeomorphically only $\operatorname{Bd} C$. Construct $\Delta$, $h$, and $h'$ as in the Mapping Theorem and its addendum with $J_2$ bounding a component of $h'(E^*_i)$ for some $i$. *Then it is easy to see that $h'$ cannot be an embedding. This is the technical difficulty that forced consideration of cell-like relations in this paper.* It is not inconceivable that the difficulties can be overcome by other means in this special case where $R$ is a function. However, if $S^{n-1}$ is replaced by a generalized $(n-1)$-manifold or if $R$ is not a function but only a cell-like embedding relation, the difficulties multiply and a technique like the Embedding Theorem is almost certainly necessary.

*Proof of the Embedding Theorem.* Because of conditions (1) and (2) of the Mapping Theorem, $(h')^{-1} | h'[C(\Delta)]$ is already continuous except possibly at points of $h'[\operatorname{Int}(E^*_0 \cup E^*_1 \cup \cdots)]$. We plan simply to split $S^n$ apart near $h'(\operatorname{Int} E^*_0)$ by a cell-like relation so as to provide enough room to isolate

$h''(\text{Int}\, E_0^*)$ from $h''[C(\Delta) - \text{Int}\, E_0^*]$. This will make $(h'')^{-1}|h''[C(\Delta)]$ continuous at points of $h''(\text{Int}\, E_0^*)$. An iteration of the splitting will serve the same purpose for $E_1^*, E_2^*, \cdots$ and complete the proof of the Embedding Theorem.

We first describe the basic splitting move as the inverse of a simple collapsing map. Define $r\colon I^2 \to [0, 1]$ by the formula $r(x) = 1/4\, d(x, \text{Bd}\, I^2) \in [0, 1/2]$. Define

$$I^2 \times_r I^{n-2} = \bigcup\{x \times [r(x)\cdot I^{n-2}] | x \in I^2\} \subset I^2 \times I^{n-2} = I^n .$$

Let $\psi\colon I^2 \times_r I^{n-2} \to (I^2 \times 0 = I^2)$ denote projection onto the first factor. If $Q$ is any neighborhood of $(\psi^{-1} \circ \psi) \cup \text{id}_{I^n}$ in $I^n \times I^n$, then $Q$ contains a PL map $\Psi\colon I^n \to I^n$ fixed on $\text{Bd}\, I^n$, extending $\psi$, and having as nondegenerate point preimages precisely the nondegenerate point preimages of $\psi$. The relation $\Psi^{-1}$ is called a *basic splitting relation.*

For each component $E$ of $h'(E_0^*)$ there is a PL embedding $P_E\colon I^2 \times I^{n-2} \to S^n$ taking $I^2 \times \{0\}$ onto $E$ and taking each fiber $x \times I^{n-2}$ onto a very small set. The embeddings $\{P_E | E \subset h'(E_0^*)\}$ may be chosen with disjoint images. Define $R_0\colon S^n \to S^n$ splitting $S^n$ along $h'(E_0^*)$ by the formula

$$R_0(x) = \begin{cases} P_E \Psi^{-1} P_E^{-1}(x) & \text{if} \quad x \in \text{Im}\, P_E \\ x & \text{if} \quad x \notin \bigcup_E \text{Im}\, P_E . \end{cases}$$

Clearly $R_0$ may be chosen in the neighborhood $N_{-1} = \varepsilon$ of $\text{id}\colon S^n \to S^n$. Define $h_0\colon C(\Delta) \to S^n$ by the formula

$$h_0(x) = \begin{cases} h'(x) & \text{if} \quad x \in E_0^* \\ R_0 \circ h'(x) & \text{if} \quad x \notin E_0^* . \end{cases}$$

Then $h_0$ is PL and injective, $R_0^{-1} \circ h_0 = h'$, and $h_0^{-1}|h_0[C(\Delta)]$ is continuous at the points of $h_0(E_0^*)$. Choose a compact neighborhood $N_0$ of $R_0$ in $\text{Int}\, N_{-1}$, slice trivial in $\text{Int}\, N_{-1}$, $N_0^{-1} \circ N_0 \subset (1)$ (Composition Theorem and [C1, Lemma 16]).

In the same manner choose $R_1\colon S^n \to S^n$ splitting $S^n$ along $h_0(E_1^*)$, fixing $R_0 \circ h'(D_0^*)$, and satisfying $R_1 \circ R_0 \subset \text{Int}\, N_0$. Define $h_1\colon C(\Delta) \to S^n$ by

$$h_1(x) = \begin{cases} h_0(x) & \text{if} \quad x \in E_1^* \\ R_1 \circ h_0(x) & \text{if} \quad x \notin E_1^* . \end{cases}$$

Choose a compact neighborhood $N_1$ of $R_1 \circ R_0$ in $\text{Int}\, N_0$, slice trivial in $\text{Int}\, N_0$, $N_1^{-1} \circ N_1 \subset (1/2)$.

In general, let $R_i$ split $S^n$ along $h_{i-1}(E_i^*)$, fixing $R_{i-1} \circ \cdots \circ R_0 \circ h'(D_{i-1}^*)$, and satisfying $R_i \circ R_{i-1} \circ \cdots \circ R_0 \subset \text{Int}\, N_{i-1}$. Define $h_i\colon C(\Delta) \to S^n$ by

$$h_i(x) = \begin{cases} h_{i-1}(x) & \text{if} \quad x \in E_i^* \\ R_i \circ h_{i-1}(x) & \text{if} \quad x \notin E_i^* . \end{cases}$$

Choose a compact neighborhood $N_i$ of $R_i \circ \cdots \circ R_0$ in Int $N_{i-1}$, slice trivial in Int $N_{i-1}$, $N_i^{-1} \circ N_i \subset 1/(i+1)$.

Define $[R' = \bigcap N_i]$: $S^n \to S^n$. As in the proof of the 1-LCC Approximation Theorem (Section 3) $R'$: $S^n \to S^n$ is a cell-like embedding relation in $\varepsilon$. Define $h'' = \bigcup_i (h_i | D_i^*)$: $C(\Delta) \to S^n$. That $h''$ is the embedding required by the Embedding Theorem is easily checked.

5.4. *The infinite Šťan'ko move.* We consider $\Delta$, $h$, and $h'$ as in the conclusion of the Mapping Theorem and its addendum. We take the relation $R'$ and the embedding $h''$ from the conclusion of the Embedding Theorem.

We identify $C(\Delta)$ with $h''C(\Delta)$ via the homeomorphism $h''$. We recall the combined identification map (*): $[D_0 \cup D_1 \cup \cdots$ (disjoint union)$] \to [D_0^* \cup D_1^* \cup \cdots = C(\Delta)] \subset S^n$.

For each $i > 0$ we identify $D_i$ from the delta symbol $(+)$ $\Delta_i = (D_i, E_i, J_i)$ with $D_i$ from a template $(+)(A_i, B_i, C_i, D_i, e_i)$ in such a manner that

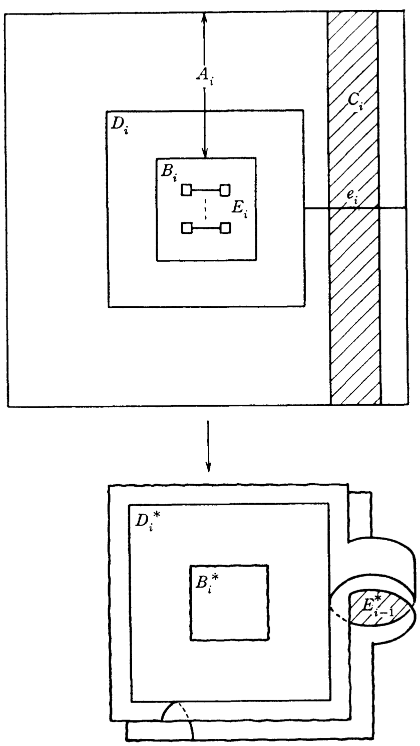


FIGURE 7. $N_i \subset Y_i \times \{0\}$.

$E_i \cup J_i \subset \text{Int } B_i$ and $(D_i \cap e_i)^* = D_i^* \cap E_{i-1}^* \subset C(\Delta) \subset S^n$. We suggest that the reader review Section 4.1 and in particular the sets $\mathcal{A}_i$, $\mathcal{B}_i$, $\mathcal{C}_i$, $D_i \times \{0\}$, and $e_i \times I$ and the homeomorphism $\Phi_n$: $\mathcal{A}_i \cup \mathcal{B}_i \to \mathcal{A}_i \cup \mathcal{B}_i$. (Figure 7.)

By the unknotting lemma of Sections 5, 6, there exist a regular neighborhood $N_i$ of $D_i^* \cup E_{i-1}^*$ in $C(\Delta)$, a PL 3-manifold $Y_i$, and a PL product $Y_i \times I^{n-3}$ in $S^n$ such that $N_i \subset Y_i \subset Y_i \times \{0\} \subset Y_i \times I^{n-3} \subset S^n$. For each $i$ we use the sets $D_i^* \cup E_{i-1}^* \subset N_i$ and the product structure $Y_i \times I^{n-3}$ to construct an embedding $\alpha_i$: $(\mathcal{A}_i \cup \mathcal{B}_i) \to S^n$ suitable for use in a basic Štan'ko move. The embedding is constructed in three steps.

Step 1. Constructing $\alpha_i | (A_i \times \{0\}) \cup (e_i \times I)$. Define $\alpha_i | (A_i \cap D_i) \times \{0\} = (*) | (A_i \cap D_i) \times \{0\}$. Since $(D_i \cap e_i)^* = D_i^* \cap E_{i-1}^*$, we may extend $\alpha_i$ to take $e_i \times I$ onto $E_{i-1}^*$ with

$$\text{Im } (R' \circ R) \cap E_{i-1}^* \subset \alpha_i[(5/4, 7/4) \times (-1, 1)] \subset \alpha_i(e_i \times I) .$$

This embedding may in turn be extended to the remainder of $A_i \times \{0\}$ so as to take $A_i - (D_i \cup e_i)$ into the $\varepsilon/i$-neighborhood (component by component) of $D_i^* \cup E_{i-1}^*$ in $Y_i - C(\Delta)$ with

$$\alpha_i(A_i \times \{0\}) \cap \text{Im } (R' \circ R) \subset \alpha_i[(5/4, 7/4) \times I] \subset \alpha_i(C_i \times \{0\}) .$$

We may require that all of the sets $\alpha_i[(A_i \times \{0\}) \cup (e_i \times I)]$ be disjoint. (Figure 7.)

*Remark*. In Steps 2 and 3 we extend our definition of $\alpha_i$ to $\mathcal{A}_i$ and to $\mathcal{B}_i$, respectively. In both steps some basic precautions can be taken. We list these precautions here.

(1) $\text{Im } (\mathcal{A}_i \cup \mathcal{B}_i) \subset (\varepsilon/i)(D_i^* \cup E_{i-1}^*)$ (component by component).

(2) Of the sets in the list $[D_0^*]$, $[\text{Im } A_1, \text{Im } \mathcal{A}_1]$, $[B_1^*, \text{Im } \mathcal{B}_1]$, $[\text{Im } A_2, \text{Im } \mathcal{A}_2^*]$, $[B_2^*, \text{Im } \mathcal{B}_2^*]$, $\cdots$ only sets in the same or adjacent square brackets can intersect.

Step 2. Constructing $\alpha_i | \mathcal{A}_i$. Then set $\mathcal{A}_i$ equals $A_i \times I \times I^{n-3}$. Since $\alpha_i(A_i \times \{0\})$ is bicollared in $Y_i \times \{0\}$, it is clearly possible to extend $\alpha_i$ to $A_i \times I$, taking each fiber $\{x\} \times I$ to a bicollar fiber $\{\alpha_i x\} \times I$ in $Y_i \times \{0\}$. In turn one may extend $\alpha_i$ to $A_i \times I \times I^{n-3}$ by taking $(x, t) \times I^{n-3}$ to $\alpha_i(x, t) \times I^{n-3}$ in the natural way. By shortening the bicollar fibers and the $I^{n-3}$ fibers of $Y_i \times I^{n-3}$ if necessary, we may protect conditions (1) and (2) of the preceding remark and obtain the following additional conditions.

(3) $$\alpha_i(\mathcal{A}_i) \cap \text{Im } (R' \circ R) \subset \alpha_i(\mathcal{C}_i) ;$$

(4) $$\alpha_i(\mathcal{A}_i \cap \Phi_n \mathcal{C}_i) \subset S^n - C(\Delta) .$$

(Condition (4) can be satisfied because of the fact that not only $D_i^* \cup E_{i-1}^*$ but