RESOLVING WILD EMBEDDINGS OF CODIMENSION-ONE MANIFOLDS IN MANIFOLDS OF DIMENSIONS GREATER THAN 3*

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For $n \geq 4$, every embedding of an $(n-1)$-manifold in an n-manifold has a δ-resolution for each $\delta > 0$. Consequently, for $n \geq 4$, every embedding of an $(n-1)$-manifold in an n-manifold can be approximated by tame embeddings.

1. Introduction

The Codimension-One Tame Approximation Theorem in dimension n states that every embedding of an $(n-1)$-manifold in an n-manifold can be approximated by tame embeddings. Bing proved this theorem in dimension 3 in [4], and went on to exploit it to great effect in his study of 3-manifolds. The theorem was established in dimensions ≥ 5 by Ancel and Cannon in [3]. It is proved in the remaining dimension, 4, in the present paper.

The Codimension-One Tame Approximation Theorem is contained in a more comprehensive proposition which is the principal result of this paper: a Resolution Theorem for Wild Codimension-One Embeddings. The latter theorem is founded on the notion of a δ-*resolution* of a wild embedding. For $\delta > 0$, a δ-resolution of a wild embedding $e: M \to N$ of a manifold M in a manifold N is, roughly speaking, a cell-like map $G: N \to N$ which moves no point of N farther than δ and to which is associated a tame embedding $f: M \to N$ such that $G \circ f = e$. Thus, the cell-like relation G^{-1} *blows up* the wild embedding $e: M \to N$ to a nearby cell-like embedding relation which *contains* the tame embedding $f: M \to N$ in the sense that $f \subset G^{-1} \circ e$. In dimension n, the essential content of the Resolution Theorem for Wild Codimension-One Embeddings is that each embedding of an $(n-1)$-manifold in an n-manifold has a δ-resolution for every $\delta > 0$.

The Resolution Theorem for Wild Codimension-One Embeddings is deduced from two other results. One of these, as might be expected, is a resolution theorem for certain generalized manifolds, which was proved in dimensions ≥ 5 in [8]. The other is an approximation theorem for cell-like maps between manifolds established in dimensions ≥ 5 in [23]. Recent work of Quinn [19] has made it possible to extend both these results to dimension 4. Consequently, the Resolution Theorem for Wild Codimension-One Embeddings as well as the Codimension-One Tame Approximation Theorem are now proved in all dimensions ≥ 4.

In its most elementary formulation, the Resolution Theorem for Wild Codimension-One Embeddings has a short and simple proof which is sketched in the next paragraph. This argument is given here to reveal the underlying ideas unobscured by technical considerations. In later sections, a variety of elaborations and generalizations of the theorem are considered. For instance, Section 6 deals with embeddings of generalized $(n-1)$-manifolds in n-manifolds, and Section 7 concerns embeddings of generalized n-manifolds with boundary in n-manifolds.

We now sketch the proof of the Resolution Theorem for Wild Codimension-One Embeddings in the simplest case. Let $n \geq 4$ and suppose M is an $(n-1)$-manifold which is embedded as a closed subset of an n-manifold N such that M separates N. Choose a metric on N and let $\delta > 0$. Let X_0 and X_1 be the closures of the components of $N - M$. Form the generalized n-manifold Y from N by *inflating* M to $M \times [0, 1]$. Thus,

$$Y = X_{0\,(M = M \times \{0\})} \cup M \times [0, 1] \cup_{(M \times \{1\} = M)} X_1.$$

Define the cell-like map $f: Y \to N$ by setting $f(\{x\} \times [0, 1]) = x$ for each $x \in M$ and letting $f|X_0 = \mathrm{id}_{X_0}$ and $f|X_1 = \mathrm{id}_{X_1}$. The nonmanifold set of Y is contained in the $(n-1)$-manifold $M \times \{0, 1\}$. There is a resolution theorem in [8], which can be extended to dimension 4 using [18, 19], and which applies to Y. It provides a cell-like map $g: P \to Y$ of an n-manifold P onto Y. The Cell-like Approximation Theorem of [23] can also be extended to dimension 4 by using [18, 19]. This theorem enables us to replace g by a *conservative* resolution. Thus, we can assume that $g: P \to Y$ is a homeomorphism over the manifold set of Y. We use the Cell-like Approximation Theorem a second time to approximate the cell-like map $f \circ g: P \to N$ by a homeomorphism $h: P \to N$, so that h is within δ of $f \circ g$. We define the cell-like map $K: N \to N$ by $K = f \circ g \circ h^{-1}$. Then K moves no point of N farther than δ. We define the embedding $j: M \to N$ by $j(x) = h \circ g^{-1}(x, \frac{1}{2})$ for each $x \in M$. j is tame because g is a homeomorphism over $M \times (0, 1)$, and $K \circ j = \mathrm{id}_M$. Thus K is a δ-resolution of the inclusion of M into N.

While still in this simple setting, we make some remarks intended to motivate the material in Section 7. Section 7 concerns the problem of approximating the inclusions of X_0 and X_1 in N by tame embeddings. Adopting the terminology of Section 7, we set

$$\mathrm{int}(X_i^+) = X_1 \cup_{(M = M \times \{0\})} M \times [0, 1)$$

for $i = 0$, 1. We assert that the inclusions of X_0 and X_1 in N can be approximated by tame embeddings if and only if $\text{int}(X_0^+)$ and $\text{int}(X_1^+)$ are n-manifolds. First assume $\text{int}(X_0^+)$ and $\text{int}(X_1^+)$ are n-manifolds. Then Y is an n-manifold. In this situation, the Cell-like Approximation Theorem implies that the cell-like map $f: Y \to N$ can be approximated by homeomorphisms. The restriction of such a homeomorphism to X_1 is a tame embedding which approximates the inclusion of X_i in N. Conversely, a tame embedding of X_i in N extends to a homeomorphism of $\text{int}(X_i^+)$ onto an open subset of N, thereby entailing that $\text{int}(X_i^+)$ be an n-manifold. Our assertion is proved.

This assertion focuses our attention on the question of whether $\text{int}(X_0^+)$ and $\text{int}(X_1^+)$ are n-manifolds. This question has been answered affirmatively in dimensions ≥ 4 by Daverman in [9, 12]. The investigations in Section 7 don't terminate with Daverman's results, because Section 7 concerns a more general situation in which M is allowed to be a *generalized* $(n-1)$-manifold. In this more general setting, the preceding question can be answered affirmatively in dimensions ≥ 5 using the theorem of Edwards in [13]. However, in this generality, the 4-dimensional version of this question remains unresolved as of this writing.

The results of this paper illustrate once more the complementary relationship between taming theory and cell-like decomposition space theory, two subjects pioneered by Bing. Our idea for the proof of the Resolution Theorem for Wild Codimension-One Embeddings was inspired by Quinn's joint use of *inflation* and *resolution* in a different context (unpublished correspondence).

We end this section with an outline of the contents of this paper. Section 2 displays definitions and statements of the main theorems of the paper as well as statements of the prerequisite theorems necessary for the proofs. These prerequisite theorems are well known in dimensions ≥ 5. Section 3 explains how results of Quinn are used to establish these prerequisite theorems in dimension 4. Sections 4 and 5 present the proof of the main theorem of this paper: the Resolution Theorem for Wild Codimension-One Embeddings of $(n-1)$-manifolds in n-manifolds in dimensions $n \geq 4$. Sections 6 and 7 explore extensions of the Resolution Theorem for Wild Codimension-One Embeddings. Specifically, Section 6 deals with embeddings of generalized $(n-1)$-manifolds in n-manifolds, and Section 7 concerns embeddings of generalized n-manifolds with boundary in n-manifolds. The results of Section 7 are not definitive in dimension 4; partial results of Daverman are described there, and a problem is posed.

2. Definitions and statements of theorems

A primary use of cell-like maps in geometric topology is to *blow up* or *resolve* singularities. Thus, cell-like maps serve to resolve generalized manifolds into topological manifolds. They can also be used to resolve wild embeddings into tame embeddings, as we now explain.

This paper is set in the topological category. Thus, the term *manifold* means *topological manifold* (*without boundary*) throughout.

An embedding $e : M \to N$ of an $(n-1)$-manifold M into an n-manifold N is *tame* if for each point x of M, there is an open neighborhood U of x in M and an embedding $E_U : U \times \mathbb{R} \to N$ such that $e(u) = E_U(u, 0)$ for every $u \in U$.

Let $e : M \to N$ be an embedding of an $(n-1)$-manifold M into an n-manifold N. The *tame set* of e, denoted $\tau(e)$, is the union of all of the open subsets U of M such that the restriction $e|U : U \to N$ is a tame embedding. The set $M - \tau(e)$ is called the *wild set* of e and is denoted $\omega(e)$.

A topological space C is *cell-like* if C is a nonempty compact metrizable space such that every map from C to an absolute neighborhood retract is homotopic to a constant map. A map $f : X \to Y$ between topological spaces is *cell-like* if f is a closed map such that $f^{-1}(y)$ is cell-like for every $y \in Y$.

Suppose $f : X \to Y$ is a map between topological spaces and $V \subset Y$. f is a *homeomorphism over* V if $f|f^{-1}(V) : f^{-1}(V) \to V$ is a homeomorphism.

Suppose $e : M \to N$ is an embedding of an $(n-1)$-manifold M into an n-manifold N, and ρ is a metric on N. Let $\delta : N \to [0, \infty)$ be a map. A δ-*resolution* of e is a cell-like map $G : N \to N$ to which is associated a tame embedding $f : M \to N$ such that $G \circ f = e$, G is a homeomorphism over $N - e(\omega(e))$, and $\rho(x, G(x)) \leq \delta(x)$ for every $x \in N$.

The following theorem is the principal result of this paper.

Theorem 2.1 (Resolution Theorem for Wild Codimension-One Embeddings). *Suppose $e : M \to N$ is an embedding of an $(n-1)$-manifold M in an n-manifold N, where $n \geq 4$. Let ρ be a metric on N. Then for every map $\delta : N \to [0, \infty)$ which is strictly positive on $e(\omega(e))$, there is a δ-resolution of e.*

As an immediate corollary, we have:

Theorem 2.2 (Codimension-One Tame Approximation Theorem). *Suppose $e : M \to N$ is an embedding of an $(n-1)$-manifold M in an n-manifold N, where $n \geq 4$. Let ρ be a metric on N. Then for every map $\delta : M \to [0, \infty)$ which is strictly positive on $\omega(e)$, there is a tame embedding $f : M \to N$ such that $\rho(e(x), f(x)) \leq \delta(x)$ for each $x \in M$.*

Proof. Given a map $\delta : M \to [0, \infty)$ which is strictly positive on $\omega(e)$, there is a map $\gamma : N \to [0, \infty)$ which is strictly positive on $e(\omega(e))$ such that $\gamma \circ e \leq \delta$. Moreover, there is a map $\beta : N \to [0, \infty)$ which is strictly positive on $e(\omega(e))$ and which has the following property: if $\rho(x, y) \leq \beta(y)$, then $\rho(x, y) \leq \gamma(x)$ for all x and y in N. Theorem 2.1 provides a β-resolution $G : N \to N$ of e. Associated with G is a tame embedding $f : M \to N$ such that $G \circ f = e$. Hence, for each $x \in M$, since

$$\rho(e(x), f(x)) = \rho(G \circ f(x), f(x)) \leq \beta(f(x)),$$

then

$$\rho(e(x), f(x)) \leq \gamma(e(x)) \leq \delta(x). \quad \square$$

As mentioned in the introduction, this theorem is valid not only in dimensions $n \geq 4$, but in dimension $n = 3$ as well, due to the work of Bing [1]. It was established in dimensions $n \geq 5$ by Ancel and Cannon [3]. The remaining dimension, $n = 4$, follows from the proof given here.

The theorems in this paper are formulated only for dimensions ≥ 4, because this is the dimension range in which the proofs given here are known to work. The two results on which the proofs rest (a resolution theorem for certain generalized manifolds and an approximation theorem for cell-like maps between manifolds) are known to be valid only in dimensions ≥ 4. (The Poincaré conjecture intervenes in dimension 3.)

We continue with the definitions needed to understand the statements of the two theorems which underlie our proofs.

Let Y be a topological space. The *manifold set* of Y, denoted $\mu(Y)$, is the union of all the open subsets of Y that are manifolds. The set $Y - \mu(Y)$ is called the *nonmanifold set* of Y and is denoted $\nu(Y)$. Thus, a point belongs to $\nu(Y)$ if and only if none of its open neighborhoods is a manifold.

A *resolution* of a topological space Y is a cell-like map $f: M \to Y$ whose domain is a manifold M and whose range is Y. A resolution $f: M \to Y$ is *conservative* if f is a homeomorphism over $\mu(Y)$.

A topological space X is a *Euclidean neighborhood retract* (ENR) if there is an embedding $e: X \to U$ of X into an open subset U of some Euclidean space $\mathbb{R}^n$ and there is a map $r: U \to X$ such that $r \circ e = \mathrm{id}_X$. Thus, a space is a Euclidean neighborhood retract if and only if it is an absolute neighborhood retract (ANR) which embeds as a closed subset of an open set in some Euclidean space. Hence, the class of ENRs coincides with the class of finite-dimensional locally compact separable ANRs.

We shall say that a topological space X is a *generalized n-manifold* if it is an ENR such that

$$H_*(X, X - \{x\}; \mathbb{Z}) \approx H_*(\mathbb{R}^n, \mathbb{R}^n - \{0\}; \mathbb{Z}) \quad \text{for every } x \in X.$$

In the early definitions of *generalized manifold* proposed by Wilder, Borel and others, a weaker homological condition appears in place of the ENR condition. In terms of this original formulation, a more appropriate name for what we have called a *generalized manifold* might be an *ENR homology manifold*. More recently, studies of cell-like decompositions of manifolds have adopted the definition given at the beginning of this section. For the sake of brevity, we shall conform to this more recent usage, acknowledging that our generalized manifolds are not as general as possible. There are a variety of equivalent definitions of *generalized manifold* in the literature; they are reconciled in [5] and in [6, Theorem 15.7].

The basic fact linking cell-like decompositions of manifolds to generalized manifolds is the following. If $f: M \to Y$ is a resolution of a finite-dimensional topological space Y, and if $\dim M = n$, then Y is a generalized n-manifold. (Use [16, Corollary 3.3] to deduce that Y is an ENR, and use [24, 25] to deduce that Y has the correct

local homology.) The converse of this fact is the *Resolution Conjecture*: every generalized n-manifold has a resolution. As of this writing, this conjecture remains open. It was supposed to have been settled affirmatively by Quinn's argument in [20] for $n \geq 5$, and then extended to $n = 4$ using [19]. However, the recent discovery of an oversight in the argument in [20] has reopened the question. Quinn corrects this oversight in [21] and recovers part of the theorem of [20]. For instance, one of the conclusions of [21] is that for $n \geq 4$, a connected generalized n-manifold has a resolution if some nonempty open subset has a resolution. Nonetheless, the Resolution Conjecture in its full generality remains open in dimensions ≥ 4. In dimension 3, even less is known about this problem.

We now state the two theorems we shall need for our proofs.

Theorem 2.3 (Resolution Theorem for Certain Generalized Manifolds). *For $n \geq 4$, a generalized n-manifold Y has a resolution if the nonmanifold set of Y is contained in a closed subset of Y which is an $(n-1)$-manifold.*

This resolution theorem was first established in dimension $n \geq 5$ in [8]. It can be extended to dimension $n = 4$ by using [18, 19]. This will be explained in Section 3.

Theorem 2.4 (Cell-like Approximation Theorem). *Suppose $f: M \to N$ is a cell-like map from an n-manifold M to an n-manifold N, where $n \geq 4$. Let ρ be a metric on N. Then for every map $\delta: N \to (0, \infty)$, there is a homeomorphism $h: M \to N$ such that $\rho(f(x), h(x)) < \delta \circ f(x)$ for every $x \in X$.*

Theorem 2.4 was proved for dimensions $n \geq 5$ in [23]. It can be extended to dimension $n = 4$ by using [18, 19]. This will be explained in Section 3.

We record a simple but useful corollary of Theorem 2.4.

Corollary 2.5. *Suppose $f: M \to Y$ is a cell-like map from an n-manifold M to a topological space Y, where $n \geq 4$; and suppose U is an open subset of $\mu(Y)$. Let ρ be a metric on Y. Then for every map $\delta: Y \to [0, \infty)$ which is strictly positive on U, there is a cell-like map $g: M \to Y$ which maps $g^{-1}(U) = f^{-1}(U)$ homeomorphically onto U and such that*

$$g = f \quad on \ g^{-1}(Y - U) = f^{-1}(Y - U)$$

and

$$\rho(f(x), g(x)) \leq \delta \circ f(x) \quad for \ every \ x \in X.$$

Proof. Le $\gamma: Y \to [0, \infty)$ be a map such that $\gamma^{-1}(0) = Y - U$ and $\gamma \leq \delta$. Theorem 2.4 provides a homeomorphism $h: f^{-1}(U) \to U$ such that

$$\rho(f(x), h(x)) < \gamma \circ f(x) \quad \text{for every } x \in f^{-1}(U).$$

Define a function $g: M \to Y$ by $g = h \cup f|f^{-1}(Y - U)$. g is continuous at points of $f^{-1}(Y - U)$ because $\gamma(Y - U) = 0$. The other required properties of g are obvious. $\square$

We mention an obvious consequence of the preceding proposition.

Corollary 2.6. *If a topological space has a resolution, then it has a conservative resolution.*

3. 4-dimensional versions of Theorems 2.3 and 2.4

The principal theorems of [18, 19] are the Controlled h-Cobordism Theorem and the Controlled End Theorem. The Controlled h-Cobordism Theorem applies to $(n+1)$-dimensional controlled h-cobordisms between n-manifolds, and the Controlled End Theorem applies to $(n+1)$-manifolds with controlled ends. In [18], these theorems are established for dimensions $n \geq 5$; and [19] deals with the case $n = 4$. We shall indicate how these two theorems lead to proofs of the 4-dimensional versions of Theorems 2.3 and 2.4.

One of the important consequences of the Controlled End Theorem is the following result.

Theorem 3.1 (Destabilization Theorem). *Let X be a generalized n-manifold, where $n \geq 4$. If $X \times \mathbb{R}$ has a resolution, then so does X.*

We recall the idea of the proof of Theorem 3.1 from [18, 19]. Let $f: N \to X \times \mathbb{R}$ be a resolution of $X \times \mathbb{R}$. Let $\pi: X \times \mathbb{R} \to X$ denote projection. Then N has two controlled ends with respect to the control map $\pi \circ f: N \to X$. In this situation, the Controlled End Theorem provides a completion $g: \bar{N} \to X$ of $\pi \circ f: N \to X$. It follows that $\bar{N}$ is an $(n+1)$-manifold with boundary, int $\bar{N} = N$, $g|N = \pi \circ f$, $\partial \bar{N}$ has two components M_0 and M_1, and both $g|M_0: M_0 \to X$ and $g|M_1: M_1 \to X$ are resolutions of X.

We are now ready to prove Theorem 2.3 for 4-manifolds.

Proof of Theorem 2.3 in dimension 4. Let Y be a generalized 4-manifold whose nonmanifold set is contained in a closed subset Z which is a 3-manifold. Then $Y \times \mathbb{R}$ is a generalized 5-manifold whose nonmanifold set is contained in the closed subset $Z \times \mathbb{R}$, and $Z \times \mathbb{R}$ is a 4-manifold. Fortuitously, the 5-dimensional version of Theorem 2.3 is proved in [8], and it provides a resolution of $Y \times \mathbb{R}$. Now Theorem 3.1 implies that Y has a resolution. □

We mention an alternative proof of this resolution theorem. In [21], it is established that, for $n \geq 4$, a connected generalized n-manifold has a resolution if some nonempty open subset has a resolution. This result covers the special type of generalized manifolds we have been considering. So Theorem 2.3 follows from the theorem in [21]. Our reason for citing [8] rather than [21] as our primary source of resolution theorems is that [8] has historical priority, and because we believe that this paper's

natural audience will find the arguments in [8] more accessible. Note, however, that
the conclusions of [21] are stronger than those of [8].

The rest of this section is devoted to a proof of Theorem 2.4 in dimension 4. Our
proof uses the following terminology and lemma.

If $\varphi : X \times [0, 1] \to Y$ is a homotopy and $\mathcal{U}$ is a cover of Y such that $\varphi(\{x\} \times [0, 1])$
is contained in an element of $\mathcal{U}$ for each $x \in X$, then φ is called a $\mathcal{U}$-*homotopy*. If
$f : X \to Y$ is a map and $\mathcal{U}$ is a cover of Y, we let

$$f^{-1}\mathcal{U} = \{f^{-1}(U) : U \in \mathcal{U}\}.$$

A map $f : X \to Y$ is a *fine homotopy equivalence* if for every open cover $\mathcal{U}$ of Y,
there is a map $g : Y \to X$, an $f^{-1}\mathcal{U}$-homotopy $\varphi : X \times [0, 1] \to X$, and a $\mathcal{U}$-homotopy
$\psi : Y \times [0, 1] \to Y$ such that $\varphi_0 = \mathrm{id}_X$, $\varphi_1 = g \circ f$, $\psi_0 = \mathrm{id}_Y$, and $\psi_1 = f \circ g$. According
to [2, 14], a cell-like map between ANRs is a fine homotopy equivalence.

Suppose $X \subset Z$. If $\mathcal{U}$ is an open cover of Z, and $\varphi : Z \times [0, 1] \to Z$ is a $\mathcal{U}$-homotopy
such that $\varphi_0 = \mathrm{id}_Z$, $\varphi_t | X = \mathrm{id}_X$ for $t \in [0, 1]$, and $\varphi_1(Z) = X$, then φ is called a $\mathcal{U}$-*strong
deformation retraction of Z onto X*. If $\pi : Z \to Y$ is a map such that for every open
cover $\mathcal{U}$ of Y, there is a $\pi^{-1}\mathcal{U}$-strong deformation retraction of Z onto X, then X
is called a *controlled strong deformation retract of Z with respect to π*.

Suppose $f : X \to Y$ is an onto map. Let $Z(f)$ denote the mapping cylinder of f.
We identify X and Y with the 'ends' of $Z(f)$ in the usual way. Let $[x, t]$ denote
the image of (x, t) under the quotient map $X \times [0, 1] \to Z(f)$. Then for each $x \in X$,
x is identified with $[x, 0]$ and $f(x)$ is identified with $[x, 1]$. Let $\pi : Z(f) \to Y$ denote
the usual mapping cylinder retraction; thus $\pi[x, t] = f(x)$ for every $[x, t] \in Z(f)$.
Consequently, $\pi | X = f$ and $\pi | Y = \mathrm{id}_Y$.

Lemma 3.2. *Suppose $f : X \to Y$ is a cell-like map between ANRs. Then X and Y are
both controlled strong deformation retracts of $Z(f)$ with respect to $\pi : Z(f) \to Y$.*

Proof. It is easy to see that Y is a controlled strong deformation retract of $Z(f)$
with respect to π. Define the homotopy $\kappa : Z(f) \times [0, 1] \to Z(f)$ by

$$\kappa([x, t], u) = [x, (1 - u)t + u] \quad \text{for } [x, t] \in Z(f), \; u \in [0, 1].$$

Then κ is a strong deformation retraction of $Z(f)$ onto Y such that $\pi \circ \kappa(\{z\} \times
[0, 1]) = \pi(z)$ for each $z \in Z(f)$. So κ is a $\pi^{-1}\mathcal{U}$-strong deformation retraction of
$Z(f)$ onto Y for each open cover $\mathcal{U}$ of Y. Also $\kappa_1 = \pi$.

The method of producing controlled strong deformation retractions of $Z(f)$ onto
X is more involved, and occupies the remainder of the proof.

Let $\mathcal{U}$ be an open cover of Y. For each open cover $\mathcal{V}$ of Y and each positive
integer k, let

$$k\mathcal{V} = \{V_1 \cup \cdots \cup V_k : V_1 \in \mathcal{V} \text{ for } 1 \leq i \leq k \text{ and } V_i \cap V_{i+1} \neq \varnothing \text{ for } 1 \leq i < k\}.$$

There is an open cover $\mathcal{V}$ of Y such that $4\mathcal{V}$ refines $\mathcal{U}$. We now invoke the fact
that $f : X \to Y$ is a fine homotopy equivalence to obtain a map $g : Y \to X$, an

$f^{-1}\mathcal{V}$-homotopy $\varphi: X \times [0, 1] \to X$, and a $\mathcal{V}$-homotopy $\psi: Y \times [0, 1] \to Y$ such that $\varphi_0 = \mathrm{id}_X$, $\varphi_1 = g \circ f$, $\psi_0 = \mathrm{id}_Y$ and $\psi_1 = f \circ g$.

The homotopy $\varphi: X \times [0, 1] \to X$ is used to define a retraction $r: Z(f) \to X$; simply set $r[x, t] = \varphi(x, t)$ for $[x, t] \in Z(f)$. r is a retraction because $r[x, 0] = \varphi(x, 0) = x = [x, 0]$ for each $x \in X$. Also $r|Y = g$, because $r[x, 1] = \varphi(x, 1) = g \circ f(x) = g[x, 1]$ for each $x \in X$. Hence, $r \circ \pi = g \circ \pi$.

Next, we produce a $\pi^{-1}2\mathcal{V}$-homotopy $\chi: Z(f) \times [0, 1] \to Z(f)$ such $\chi_0 = \mathrm{id}_{Z(f)}$ and $\chi_1 = r$. χ breaks naturally into four distinct stages.

$$\chi_t = \begin{cases} \kappa_{4t} & \text{for } t \in [0, \tfrac{1}{4}], \\ \psi_{4t-1} \circ \pi & \text{for } t \in [\tfrac{1}{4}, \tfrac{1}{2}], \\ \kappa_{3-4t} \circ g \circ \pi & \text{for } t \in [\tfrac{1}{2}, \tfrac{3}{4}], \\ r \circ \kappa_{4-4t} & \text{for } t \in [\tfrac{3}{4}, 1]. \end{cases}$$

χ is well defined because $\kappa_1 = \pi = \psi_0 \circ \pi$, $\psi_1 \circ \pi = f \circ g \circ \pi = \pi \circ g \circ \pi = \kappa_1 \circ g \circ \pi$, and $\kappa_0 \circ g \circ \pi = g \circ \pi = r \circ \pi = r \circ \kappa_1$. So $\chi_0 = \kappa_0 = \mathrm{id}_{Z(f)}$ and $\chi_1 = r \circ \kappa_0 = r$. During each of the time periods $[0, \tfrac{1}{4}]$ and $[\tfrac{1}{2}, \tfrac{3}{4}]$, π maps each track of χ to a point in Y; while during each of the time periods $[\tfrac{1}{4}, \tfrac{1}{2}]$ and $[\tfrac{3}{4}, 1]$, π maps each track of χ into an element of $\mathcal{V}$. Thus, χ is a $\pi^{-1}2\mathcal{V}$-homotopy from $\mathrm{id}_{Z(f)}$ to the retraction $r: Z(f) \to X$.

Unfortunately, χ is not a *strong* deformation retraction because it fails to fix the points of X. The following proposition remedies this failure.

Proposition 3.3. *Suppose $r: Z \to A$ is a retraction map from an ANR Z onto a closed subset A ($r|A = \mathrm{id}_A$), $\mathcal{W}$ is an open cover of Z, and $\chi: Z \times [0, 1] \to Z$ is a $\mathcal{W}$-homotopy such that $\chi_0 = \mathrm{id}_Z$ and $\chi_1 = r$. Then there is a $2\mathcal{W}$-homotopy $\omega: Z \times [0, 1] \to Z$ such that $\omega_0 = \mathrm{id}_Z$, $\omega_1 = r$, and $\omega_t|A = \mathrm{id}_A$ for $t \in [0, 1]$.*

The uncontrolled version of this proposition (with no mention of an open cover $\mathcal{W}$) appears as [15, Theorem 2.1]. The proof given there also yields a proof of the proposition stated here, if one pays attention to the tracks of homotopies and uses the following controlled version of Borsuk's Homotopy Extension Principle at the appropriate point.

Proposition 3.4 (Controlled Borsuk Homotopy Extension Principle). *Suppose C is a closed subset of a metrizable space T, Z is an ANR, $\chi: (T \times \{0\}) \cup (C \times [0, 1]) \to Z$ is a map, and $\mathcal{W}$ is an open cover of Z such that $\chi|C \times [0, 1]$ is a $\mathcal{W}$-homotopy. Then χ extends to a $\mathcal{W}$-homotopy $\bar{\chi}: T \times [0, 1] \to Z$.*

With a little care, the usual proof of the Borsuk Homotopy Extension Principle [15, Theorem 2.2] can be modified to a proof of the controlled version.

$Z(f)$ is an ANR because f is a proper map between ANRs [15, Theorem 1.2]. Given the $\pi^{-1}2\mathcal{V}$-homotopy $\chi: Z(f) \times [0, 1] \to Z(f)$ from $\mathrm{id}_{Z(f)}$ to the retraction

$r: Z(f) \to X$, the above proposition provides a $\pi^{-1}4\mathcal{V}$-homotopy $\omega: Z(f) \times [0, 1] \to Z(f)$ from $\mathrm{id}_{Z(f)}$ to r such that $\omega_t|X = \mathrm{id}_X$ for all $t \in [0, 1]$. Thus, ω is a $\pi^{-1}\mathcal{U}$-strong deformation retraction of $Z(f)$ onto X. This completes the proof that X is a controlled strong deformation retract of $Z(f)$ with respect to π. $\square$

Proof of Theorem 2.4 in dimension 4. Suppose $f: M \to N$ is a cell-like map between 4-manifolds. Our proof has three steps. First, we prove that $Z(f)$ is a manifold. Then we argue that $Z(f)$ is a controlled h-cobordism with respect to the map $\pi: Z(f) \to N$. Last, we invoke the Controlled h-Cobordism Theorem to obtain a homeomorphism from M to N approximating f.

Let ρ be a metric on N. Define the metric σ on $N \times [0, 1]$ by

$$\sigma((y_1, t_1), (y_2, t_2)) = \max\{\rho(y_1, y_2), |t_1 - t_2|\}.$$

Consider the cell-like map between 5-manifolds

$$f \times \mathrm{id}_{(0,1)}: M \times (0, 1) \to N \times (0, 1).$$

By virtue of [23], Theorem 2.4 holds in dimensions $\geqslant 5$. Hence, there is a homeomorphism $G: M \times (0, 1) \to N \times (0, 1)$ such that

$$\sigma((f(x), t), G(x, t)) < 1 - t \quad \text{for all } (x, t) \in M \times (0, 1).$$

It follows that G extends to a map $\bar{G}: M \times (0, 1] \to N \times (0, 1]$ such that $\bar{G}(x, 1) = (f(x), 1)$ for all $x \in M$. Let $Z = M \times [0, 1) \cup_G N \times (0, 1]$. Clearly, Z is a 5-manifold with boundary, and ∂Z is homeomorphic to the disjoint union of M and N. Apparently, Z is homeomorphic to $M \times [0, 1] \cup_{\bar{G}} N \times (0, 1]$, and the latter space is clearly homeomorphic to the mapping cylinder $Z(f)$. Thus, $Z(f)$ is a 5-manifold with boundary, and $\partial Z(f)$ is the union of the two ends, M and N, of $Z(f)$.

To prove that $Z(f)$ is a controlled h-cobordism with respect to the map $\pi: Z(f) \to N$, we must produce, for each map $\gamma: N \to (0, \infty)$, strong deformation retractions of $Z(f)$ onto M and N with track-size bounded by γ in the following sense. Under each deformation, the track of every point $z \in Z(f)$ is mapped by π to a set of diameter $< \gamma \circ \pi(z)$. Suppose we are given a map $\gamma: N \to (0, \infty)$. There is an open cover $\mathcal{U}$ of N such that for each $U \in \mathcal{U}$, diam $U < \gamma(y)$ for every $y \in U$. Lemma 3.2 provides $\pi^{-1}\mathcal{U}$-strong deformation retractions of $Z(f)$ onto M and N. Clearly, these deformations have track-size bounded by γ in the sense just mentioned.

Let $\delta: N \to (0, \infty)$ be a map. Since $Z(f)$ is a 5-dimensional controlled h-cobordism with respect to the control map $\pi: Z(f) \to N$, then the Controlled h-Cobordism Theorem of [19] provides a homeomorphism $H: M \times [0, 1] \to Z(f)$ such that

$$H(x, 0) = [x, 0] \quad \text{and} \quad \rho(\pi \circ H(x, t), \pi[x, 0]) < \delta \circ \pi[x, 0]$$

$$\text{for all } x \in M, \ t \in [0, 1].$$

Thus, H maps $M \times \{1\}$ homeomorphically onto N. Since $\pi|N = \mathrm{id}_N$, then a homeomorphism $h: M \to N$ is defined by $h(x) = \pi \circ H(x, 1)$ for $x \in M$. Recall that $\pi[x, 0] = f(x)$ for each $x \in M$. Therefore

$$\rho(h(x), f(x)) < \delta \circ f(x) \quad \text{for each } x \in M. \quad \square$$

4. The proof of Theorem 2.1 in a special case

Our proof requires the following definition and lemma. A space X is a *generalized n-manifold with boundary* if X is an ENR which has a closed subset, denoted ∂X, such that ∂X is a generalized $(n-1)$-manifold, $\text{int}(X) = X - \partial X$ is a generalized n-manifold, and $H_*(X, X - \{x\}) = 0$ for each $x \in \partial X$. ∂X is called the *boundary* of X, and $\text{int}(X)$ is called the *interior* of X.

Lemma 4.1. *Suppose that X_0, X_1 and Z are closed subsets of a topological space Y such that $X_0 \cup X_1 = Y$ and $X_0 \cap X_1 = Z$. Then, Y is a generalized n-manifold and Z is a generalized $(n-1)$-manifold if and only if X_0 and X_1 are both generalized n-manifolds with boundary equal to Z.*

Proof. First, assume that Y is a generalized n-manifold and Z is a generalized $(n-1)$-manifold. Hu [15, Proposition 9.1] implies that X_0 and X_1 are ENRs, and Raymond [22, Theorem 2] reveals that X_0 and X_1 have the correct local homology to be generalized n-manifolds with boundary equal to Z.

Second, assume that X_0 and X_1 are generalized n-manifolds with boundary equal to Z. Hu [15, Proposition 10.1] implies that Y is an ENR, and Raymond [22, Theorem 4] reveals that Y has the correct local homology to be a generalized n-manifold. $\square$

Now suppose $e: M \to N$ is an embedding of an $(n-1)$-manifold M in an n-manifold N, where $n \geq 4$. In this section we make the following simplifying assumption: $e(M)$ is a closed subset of N which separates N into exactly two components, and $e(M)$ is the frontier of each component of $N - e(M)$.

Brown [7] provides a *bicollar* on $e(\tau(e))$ in N. By restricting this bicollar and tapering it near $e(\omega(e))$, we obtain a closed map $c: M \times [0, 1] \to N$ such that $c(x, \frac{1}{2}) = e(x)$ for each $x \in M$, $c|\tau(e) \times [0, 1]$ is an embedding, and $c^{-1}(e(x)) = \{x\} \times [0, 1]$ for every $x \in \omega(e)$. Then $c(M \times [0, 1])$ separates N into exactly two components. Let X_0 and X_1 be the closures of the two components of $N - c(M \times [0, 1])$ such that $c(M \times \{0\}) \subset X_0$ and $c(M \times \{1\}) \subset X_1$. Lemma 4.1 reveals that X_0 and X_1 are generalized n-manifolds with boundary, and that $\partial X_0 = c(M \times \{0\})$ and $\partial X_1 = c(M \times \{1\})$.

Set
$$Y = X_0 \,_{(c|M \times \{0\})} \cup \, M \times [0, 1] \,\cup_{(c|M \times \{1\})} X_1.$$

In other words, to obtain Y, remove $c(M \times [0, 1])$ from N and sew in $M \times [0, 1]$, using $c|M \times \{0\}$ to attach $M \times \{0\}$ to ∂X_0, and using $c|M \times \{1\}$ to attach $M \times \{1\}$ to ∂X_1. Lemma 4.1 implies that Y is a generalized n-manifold.

Define the map $f: Y \to N$ by $f(x, t) = c(x, t)$ for each $(x, t) \in M \times [0, 1]$, $f|X_0 = \text{id}_{X_0}$ and $f|X_1 = \text{id}_{X_1}$. f is a cell-like map, because $f^{-1}(x)$ is an arc for each $x \in e(\omega(e))$, and $f^{-1}(x)$ is a point for each $x \in N - e(\omega(e))$.

Observe that the nonmanifold set of Y, $\nu(Y)$, lies in $\omega(e) \times \{0, 1\}$. Hence $\nu(Y)$ is contained in the $(n-1)$-manifold $M \times \{0, 1\}$ which is a closed subset of Y. Consequently, the Theorem 2.3 provides a cell-like map $g : P \to Y$ from an n-manifold P onto Y. Moreover, Corollary 2.6 allows us to assume that $g : P \to Y$ is a conservative resolution.

Since the composition of cell-like maps is cell-like (an immediate consequence of [16, Theorem 1.4]), then $f \circ g : P \to N$ is a cell-like map. Since $g : P \to Y$ is conservative, it is a homeomorphism over $Y - (\omega(e) \times \{0, 1\}) = f^{-1}(N - e(\omega(e)))$. Also f is a homeomorphism over $N - e(\omega(e))$. Consequently, $f \circ g$ is a homeomorphism over $N - e(\omega(e))$.

Suppose ρ is a metric on N, and $\delta : N \to [0, \infty)$ is a map which is strictly positive on $e(\omega(e))$. There is a map $\gamma : N \to [0, \infty)$ which is strictly positive on $e(\omega(e))$ and which has the following property: if $\rho(x, y) \leq \gamma(y)$, then $\rho(x, y) \leq \delta(x)$ for all x and y in N. Set $U = \gamma^{-1}(0, \infty)$. Since $e(\omega(e)) \subset U$, then $f \circ g$ is a homeomorphism over $N - U$.

Now, we invoke Theorem 2.4 to obtain a homeomorphism $h_U : (f \circ g)^{-1}(U) \to U$ such that

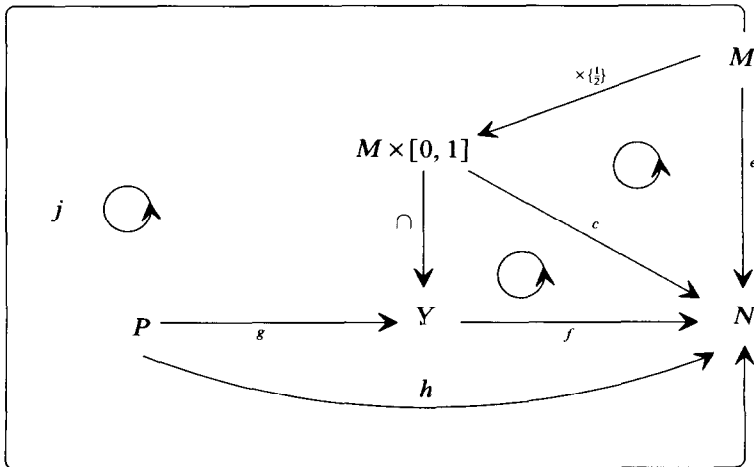$$\rho(f \circ g(x), h_U(x)) < \gamma \circ f \circ g(x) \quad \text{for every } x \in (f \circ g)^{-1}(U).$$

Define the function $h : P \to N$ by

$$h = h_U \cup f \circ g|(f \circ g)^{-1}(N - U).$$

Since $\gamma = 0$ on $N - U$, then h is continuous. Since $f \circ g$ is a homeomorphism over $N - U$, then $h : P \to N$ is a homeomorphism such that

$$\rho(f \circ g(x), h(x)) \leq \gamma \circ f \circ g(x) \quad \text{for every } x \in P.$$

Define the cell-like map $K : N \to N$ by $K = f \circ g \circ h^{-1}$. We shall prove that K is a δ-resolution of $e : M \to N$. First, we define the map $j : M \to N$ by $j(x) = h \circ g^{-1}(x, \frac{1}{2})$

for each $x \in M$. Since g^{-1} embeds $M \times (0, 1)$ in P and $h : P \to N$ is a homeomorphism, then $j : M \to N$ is a tame embedding. $K \circ j = e$ because

$$K \circ j(x) = f \circ g \circ h^{-1} \circ h \circ g^{-1}(x, \tfrac{1}{2})$$
$$= f(x, \tfrac{1}{2}) = c(x, \tfrac{1}{2}) = e(x) \quad \text{for each } x \in M.$$

Since $f \circ g$ is a homeomorphism over $N - e(\omega(e))$, and since $h : P \to N$ is a homeomorphism, then K is a homeomorphism over $N - e(\omega(e))$. Finally, for each $x \in N$,

$$\rho(x, K(x)) = \rho(h \circ h^{-1}(x), f \circ g \circ h^{-1}(x))$$
$$\leq \gamma \circ f \circ g \circ h^{-1}(x) = \gamma \circ K(x).$$

Hence, $\rho(x, K(x)) \leq \delta(x)$ for each $x \in N$. $\quad \square$

5. The proof of Theorem 2.1 in the general case

The proof in this section is a rather technical convergence argument. We make repeated local applications of the special case of Theorem 2.1 established in the previous section. We thereby obtain a sequence of cell-like maps whose limit is the sought-after δ-resolution.

We begin with some useful general information. Recall that a map $f : X \to Y$ is *proper* if $f^{-1}(C)$ is compact for every compact subset C of Y. A map $f : X \to Y$ between metrizable spaces is proper if and only if f is a closed map such that $f^{-1}(y)$ is compact for each $y \in Y$. A metric σ on a space Y is *proper* if it has the property that a closed subset of Y is compact if and only if its σ-diameter is finite. The following two facts establish a connection between proper metrics and proper maps.

(1) If ρ is a metric on a locally compact and σ-compact space Y, then there is a proper metric σ on Y such that $\rho \leq \sigma$. Simply set

$$\sigma(x, y) = \rho(x, y) + |\varphi(x) - \varphi(y)|$$

where $\varphi : Y \to [0, \infty)$ is a proper map.

(2) If σ is a proper metric on a space Y, $f : Y \to Y$ is a map, and r is a constant such that $\sigma(y, f(y)) \leq r$ for each $y \in Y$, then f is a proper map. Indeed, if C is a compact subset of Y, then $f^{-1}(C)$ is compact because

$$\sigma\text{-diam}(f^{-1}(C)) \leq \sigma\text{-diam}(C) + 2r.$$

Finally, we observe that a proper metric is complete.

Suppose $e : M \to N$ is an embedding of an $(n-1)$-manifold M in an n-manifold N, where $n \geq 4$. Suppose ρ is a metric on N. Let $\delta : N \to [0, \infty)$ be a map which is strictly positive on $e(\omega(e))$.

We can assume that ρ is a proper metric on N. Indeed, there is a proper metric σ on N such that $\rho \leq \sigma$. Clearly, a δ-resolution of e with respect to the metric σ is also a δ-resolution of e with respect to the metric ρ. So ρ can be replaced by σ, if

necessary. Also, we can assume $\delta \leq 1$; because otherwise δ can be replaced by $\min\{\delta, 1\}$.

A connected open subset W of M is called an *absolute separator* if it has the following property. If $f : M \to N$ is any embedding and V is any connected open subset of W, then there is a connected open subset U of N such that $f(M) \cap U = f(V)$, $f(V)$ is a (relatively) closed subset of U which separates U into exactly two components, and $f(V)$ is the (relative) frontier in U of each component of $U - f(V)$. Observe that any connected open subset of an absolute separator is itself an absolute separator. It is a fact that M is covered by absolute separators. Indeed, if W and W^* are connected open subsets of M with compact closures such that $\mathrm{cl}(W)$ contracts to a point in W^* and $\mathrm{cl}(W^*)$ contracts to a point in M, then W is an absolute separator. (See [1, Theorem VI.4 and Proposition XV.12].)

Let $V_0 = M - \omega(e)$. There is a locally finite open cover $\{V_i : i \geq 0\}$ of M such that for each $i \geq 1$, V_i is an absolute separator lying in $e^{-1}(\delta^{-1}(0, \infty))$. For each $i \geq 0$, set $D_i = M - (\bigcup_{j > i} V_j)$. Then each D_i is a closed subset of M, $D_0 \subset V_0$, $D_i \subset D_{i-1} \cup V_i$ for each $i \geq 1$, and $\{\mathrm{int}(D_i) : i \geq 0\}$ covers M.

Set $\delta_0 = \delta$, $G_0 = \mathrm{id}_N$ and $f_0 = e$. We shall construct three sequences: a sequence of maps $\delta_i : N \to [0, \infty)$, a sequence of cell-like maps $G_i : N \to N$, and a sequence of embeddings $f_i : M \to N$ with the following properties.

(1) $\delta_i \leq 2^{-i}\delta$. Furthermore, for $0 \leq j \leq i - 1$ and for all $x, y \in N$, if $\rho(x, y) \leq \delta_i(x)$, then

$$\rho(G_j \circ \cdots \circ G_{i-1}(x), G_j \circ \cdots \circ G_{i-1}(y)) \leq 2^{-i}\delta(x).$$

(2) $\rho(x, G_i(x)) \leq \delta_i(x)$ for each $x \in N$.

(3) $G_i \circ f_i = f_{i-1}$.

(4) G_i is a homeomorphism over $N - f_{i-1}(\omega(f_{i-1}))$.

(5) $f_i | D_{i-1} = f_{i-1} | D_{i-1}$.

(6) $\omega(f_i) \subset \omega(f_{i-1}) - D_i$.

The construction of $\{\delta_i\}$, $\{G_i\}$ and $\{f_i\}$ proceeds by induction. Suppose $i \geq 1$, and assume we have δ_j, G_j, and f_j for $0 \leq j < i$ satisfying properties (1) through (6).

Since V_i is an absolute separator, there is a connected open subset U of N such that $f_{i-1}(M) \cap U = f_{i-1}(V_i)$, $f_{i-1}(V_i)$ is a closed subset of U which separates U into exactly two components, and $f_{i-1}(V_i)$ is the (relative) frontier in U of each component of $U - f_{i-1}(V_i)$. We shall apply the special case of Theorem 2.1 established in Section 4 to the embedding $f_{i-1} | V_i : V_i \to U$. The secret to achieving the desired outcome is in choosing δ_i correctly.

First we observe that $f_{i-1}(V_i) \subset \delta^{-1}(0, \infty)$. We argue that, in fact, $V_i \cap f_j^{-1}(\delta^{-1}(0)) = \varnothing$ for $0 \leq j < i$. This is clear for $j = 0$. For $1 \leq j < i$, if $x \in f_j^{-1}(\delta^{-1}(0))$, then $\rho(f_j(x), f_{i-1}(x)) = \rho(f_j(x), G_j \circ f_j(x)) \leq \delta_j(f_j(x)) \leq 2^{-j}\delta(f_j(x)) = 0$, making $f_{i-1}(x) = f_j(x)$. Consequently, $f_j^{-1}(\delta^{-1}(0)) \subset f_{i-1}(\delta^{-1}(0))$ for $1 \leq j < i$. Thus, $V_i \cap f_{i-1}^{-1}(\delta^{-1}(0)) = \varnothing$ implies that $V_i \cap f_j^{-1}(\delta^{-1}(0)) = \varnothing$ for $1 \leq j < i$.

Our second step is to produce an open subset $\tilde{R}$ of $U \cap \delta^{-1}(0, \infty)$ such that $f_{i-1}(\omega(f_{i-1} | V_i)) \subset \tilde{R}$ and $f_{i-1}^{-1}(\mathrm{cl}(\tilde{R})) \subset (V_i \cup \omega(f_{i-1})) - D_{i-1}$. Set $Z = \omega(f_{i-1}) - V_i$. Z

is a closed subset of M. (6) implies that $D_{i-1} \cap \omega(f_{i-1}) = \varnothing$. Hence, $((M - V_i) \cup D_{i-1}) \cap \omega(f_{i-1}) = Z$. Consequently, in the subspace $M - Z$, $((M - V_i) \cup D_{i-1}) - Z$ and $\omega(f_{i-1}) - Z$ are disjoint (relatively) closed sets. So there are disjoint open subsets Q and R of $M - Z$ such that $((M - V_i) \cup D_{i-1}) - Z \subset Q$ and $\omega(f_{i-1}) - Z \subset R$. Since $D_{i-1} \cap Z = \varnothing$, it follows that $D_{i-1} \subset Q$; also we note that $\{V_i, Q, Z\}$ covers M, and that $R \cap (Q \cup Z) = \varnothing$. From these three facts, we deduce that $M - Q \subset (V_i \cup \omega(f_{i-1})) - D_{i-1}$ and that $R \subset V_i$. Since $f_{i-1}(Q)$ and $f_{i-1}(R)$ are disjoint (relatively) open subsets of $f_{i-1}(M)$, and since $f_{i-1}(R) \subset f_{i-1}(V_i) \subset U \cap \delta^{-1}(0, \infty)$, then there are disjoint open subsets $\tilde{Q}$ and $\tilde{R}$ of N such that $f_{i-1}(Q) = f_{i-1}(M) \cap \tilde{Q}$, $f_{i-1}(R) = f_{i-1}(M) \cap \tilde{R}$ and $\tilde{R} \subset U \cap \delta^{-1}(0, \infty)$. It follows that $f_{i-1}(\omega(f_{i-1}|V_i)) \subset \tilde{R}$, because $\omega(f_{i-1}|V_i) = \omega(f_{i-1}) \cap V_i = \omega(f_{i-1}) - Z \subset R = f_{i-1}^{-1}(\tilde{R})$. Since $\tilde{Q} \cap \mathrm{cl}(\tilde{R}) = \varnothing$, then $Q \cap f_{i-1}^{-1}(\mathrm{cl}(\tilde{R})) = \varnothing$. Consequently, $f_{i-1}^{-1}(\mathrm{cl}(\tilde{R})) \subset M - Q \subset (V_i \cup \omega(f_{i-1})) - D_{i-1}$. Thus, $\tilde{R}$ has the desired properties.

We now choose the map $\delta_i : N \to [0, \infty)$ so that $\delta_i^{-1}(0, \infty) = \tilde{R}$ and $\delta_i(x) \leqslant (\frac{1}{2})\rho(x, N - \tilde{R})$ for each $x \in N$, and so that δ_i satisfies property (1). This is possible because $\tilde{R} \subset \delta^{-1}(0, \infty)$ and because δ and $G_j \circ \cdots \circ G_{i-1}$ are continuous for $0 \leqslant j \leqslant i - 1$.

Since $f_{i-1}(\omega(f_{i-1}|V_i)) \subset \delta_i^{-1}(0, \infty)$, we can apply the special case of the Theorem 2.1 to the embedding $f_{i-1}|V_i : V_i \to U$. In this way, we obtain a $(\delta_i|U)$-resolution $\Gamma : U \to U$ of $f_{i-1}|V_i : V_i \to U$. Let $\varphi : V_i \to U$ be the tame embedding associated with Γ; thus $\Gamma \circ \varphi = f_{i-1}|V_i$.

Since $\rho(x, \Gamma(x)) \leqslant \delta_i(x)$ for each $x \in U$, and since $\delta_i = 0$ on $N - U$, then a map $G_i : N \to N$ is defined by $G_i = \Gamma \cup \mathrm{id}_{N-U}$. Clearly, $G_i : N \to N$ is a cell-like map satisfying $\rho(x, G_i(x)) \leqslant \delta_i(x)$ for each $x \in N$. Define the function $f_i : M \to N$ by $f_i = \varphi \cup (f_{i-1}|M - V_i)$. Then clearly $G_i \circ f_i = f_{i-1}$.

We shall now prove that f_i is continuous. It suffices to consider a sequence $\{x_j\}$ in V_i which converges to a point $x \in M - V_i$, and to prove that $\{f_i(x_j)\}$ converges to $f_i(x)$. Since $\{f_{i-1}(x_j)\}$ converges to $f_{i-1}(x)$, and $f_{i-1}(x) = f_i(x)$, then it clearly suffices to show that $\{\rho(f_i(x_j), f_{i-1}(x_j))\}$ converges to 0. Note first that

$$\rho(f_i(x_j), f_{i-1}(x_j)) = \rho(f_i(x_j), G_i \circ f_i(x_j)) \leqslant \delta_i(f_i(x_j))$$

$$\leqslant (\tfrac{1}{2})\rho(f_i(x_j), N - \tilde{R}) \leqslant (\tfrac{1}{2})\rho(f_i(x_j), N - U).$$

Hence,

$$\rho(f_i(x_j), N - U) \leqslant \rho(f_i(x_j), f_{i-1}(x_j)) + \rho(f_{i-1}(x_j), N - U)$$

$$\leqslant (\tfrac{1}{2})\rho(f_i(x_j), N - U) + \rho(f_{i-1}(x_j), N - U).$$

So,

$$(\tfrac{1}{2})\rho(f_i(x_j), N - U) \leqslant \rho(f_{i-1}(x_j), N - U).$$

Combining the first and third of these inequalities, we have

$$\rho(f_i(x_j), f_{i-1}(x_j)) \leqslant \rho(f_{i-1}(x_j), N - U).$$

Now $f_{i-1}(x) \in N - U$, because $x \in M - V_i$. Since $\{f_{i-1}(x_j)\}$ converges to $f_{i-1}(x)$, then the sequence $\{\rho(f_{i-1}(x_j), N - U)\}$ converges to 0. We conclude that the sequence $\{\rho(f_i(x_j), f_{i-1}(x_j))\}$ converges to 0.

Now that we know that $f_i : M \to N$ is continuous, we easily deduce that it is an embedding from the equation $G_i \circ f_i = f_{i-1}$ and the fact that $f_{i-1} : M \to N$ is an embedding.

Since $\Gamma : U \to U$ is a homeomorphism over $U - f_{i-1}(\omega(f_{i-1}|V_i))$, then G_i is a homeomorphism over $(U - f_{i-1}(\omega(f_{i-1}|V_i))) \cup (N - U)$. Thus, G_i is a homeomorphism over $N - f_{i-1}(\omega(f_{i-1}))$.

Now we shall prove that $f_i = f_{i-1}$ on $M - f_{i-1}^{-1}(\tilde{R})$. Let $x \in M - f_{i-1}^{-1}(\tilde{R})$. Then $f_{i-1}(x) \in N - \tilde{R}$. So

$$\rho(f_i(x), N - \tilde{R}) \leq \rho(f_i(x), f_{i-1}(x)) = \rho(f_i(x), G_i \circ f_i(x))$$

$$\leq \delta_i(f_i(x)) \leq (\tfrac{1}{2})\rho(f_i(x), N - \tilde{R}).$$

Since $\rho(f_i(x), N - \tilde{R}) = 0$. It follows that $f_i(x) \in N - \tilde{R}$ and that $\delta_i(f_i(x)) = 0$. Since $\rho(f_i(x), f_{i-1}(x)) \leq \delta_i(f_i(x))$, we conclude that $f_i(x) = f_{i-1}(x)$.

Since $f_{i-1}^{-1}(\tilde{R}) \cap D_{i-1} = \varnothing$, then the result of the previous paragraph implies that $f_i|D_{i-1} = f_{i-1}|D_{i-1}$.

$f_i|V_i$ is tame because $f_i|V_i = \varphi$. Since $f_i = f_{i-1}$ on $M - f_{i-1}^{-1}(\text{cl}(\tilde{R}))$, then f_i is also tame on $M - (f_{i-1}^{-1}(\text{cl}(\tilde{R})) \cup \omega(f_{i-1}))$. Consequently, $\omega(f_i) \subset (f_{i-1}^{-1}(\text{cl}(\tilde{R})) \cup \omega(f_{i-1})) - V_i$. Since $f_{i-1}^{-1}(\text{cl}(\tilde{R})) \subset V_i \cup \omega(f_{i-1})$, we conclude that $\omega(f_i) \subset \omega(f_{i-1}) - V_i$. $\omega(f_{i-1}) - V_i \subset \omega(f_{i-1}) - D_i$, because $D_i \subset D_{i-1} \cup V_i$ and $D_{i-1} \cap \omega(f_{i-1}) = \varnothing$. Therefore, $\omega(f_i) \subset \omega(f_{i-1}) - D_i$.

We have now completed the verification that δ_i, G_i and f_i satisfy properties (1) through (6). Hence, the sequences $\{\delta_i\}$, $\{G_i\}$ and $\{f_i\}$ can be constructed as desired.

We now argue that for each $i \geq 0$, the sequence $\{G_i \circ G_{i+1} \circ \cdots \circ G_j : j \geq i\}$ converges to a cell-like map $H_i : N \to N$ such that

$$\rho(G_i \circ G_{i+1} \circ \cdots \circ G_j(x), H_i(x)) \leq 2^{-j}\delta(x) \quad \text{for each } x \in N.$$

Indeed, properties (1) and (2) imply that

$$\rho(G_i \circ \cdots \circ G_{j-1}(x), G_i \circ \cdots \circ G_{j-1} \circ G_j(x)) \leq 2^{-j}\delta(x) \leq 2^{-j}$$

for each $x \in N$. Hence, the map $H_i : N \to N$ exists as asserted. Furthermore, $\rho(x, H_i(x)) \leq 2^{-i+1}$ for each $x \in N$; indeed, $\rho(x, H_i(x)) \leq \rho(x, G_i(x)) + \rho(G_i(x), H_i(x)) \leq \delta_i(x) + 2^{-i}\delta(x) \leq 2 \cdot 2^{-i}\delta(x) \leq 2^{-i+1}$ for each $x \in N$. Since ρ is a proper metric, then H_i is a proper map. Thus, H_i is a closed map which is the uniform limit of surjections; this forces H_i to be a surjection. Each composition $G_i \circ G_{i+1} \circ \cdots \circ G_j$ is cell-like as a consequence of [16, Theorem 1.4]. So H_i is a proper surjection which is the limit of cell-like maps. Now [17, Theorem 3.1] implies that H_i is a cell-like map. We note that for $i \leq j$, since the sequence $\{G_i \circ G_{i+1} \circ \cdots \circ G_k : k \geq j\}$ converges to both H_i and $G_i \circ \cdots \circ G_{j-1} \circ H_j$, then $H_i = G_i \circ \cdots \circ G_{j-1} \circ H_j$. We are most interested in the cell-like map $H_0 : N \to N$. So we set $H = H_0$. Then, $\rho(x, H(x)) = \rho(G_0(x), H_0(x)) \leq \delta(x)$ for each $x \in N$.

Next we observe that $\{f_i\}$ converges uniformly to a map $f: M \to N$. Indeed, properties (1), (2) and (3) imply that $\rho(f_i(x), f_{i-1}(x)) = \rho(f_i(x), G_i \circ f_i(x)) \leq \delta_i(f_i(x)) \leq 2^{-i} \delta(f_i(x)) \leq 2^{-i}$ for each $x \in M$. From property (5), we deduce that $f_j|D_i = f_i|D_i$ for each $j \geq i$. Hence, $f|D_i = f_i|D_i$ for each $i \geq 1$. Property (3) implies that $G_0 \circ \cdots \circ G_i \circ f_i = e$ for each $i \geq 1$. Consequently, for each $i \geq 1$, $G_0 \circ \cdots \circ G_i \circ f|D_i = G_0 \circ \cdots \circ G_i \circ f_i|D_i = e|D_i$. Thus, $H \circ f|D_i = e|D_i$ for each $i \geq 1$. Since $\cup_{i \geq 1} D_i = M$, we have $H \circ f = e$. Since e is an embedding, it follows immediately that f is an embedding. Property (6) implies that each $f_i|\text{int}(D_i)$ is tame. Hence, each $f|\text{int}(D_i)$ is tame. Since $\{\text{int}(D_i): i \geq 0\}$ covers M, we conclude that f is tame.

Our final task is to prove that H is a homeomorphism over $N - e(\omega(e))$. We begin this task by establishing that $G_0 \circ \cdots \circ G_i$ is a homeomorphism over $N - e(\omega(e))$. This is clear for $i = 0$. Let $i \geq 1$ and inductively assume that $G_0 \circ \cdots \circ G_{i-1}$ is a homeomorphism over $N - e(\omega(e))$. Properties (3) and (6) imply that $G_j \circ f_j(\omega(f_j)) \subset f_{j-1}(\omega(f_{j-1}))$ for each $j \geq 1$. Hence, $G_0 \circ \cdots \circ G_{i-1} \circ f_{i-1}(\omega(f_{i-1})) \subset e(\omega(e))$. Consequently, $(G_0 \circ \cdots \circ G_{i-1})^{-1}(N - e(\omega(e))) \subset N - f_{i-1}(\omega(f_{i-1}))$. It now follows from property (4) that G_i is a homeomorphism over $(G_0 \circ \cdots \circ G_{i-1})^{-1}(N - e(\omega(e)))$. With the help of the inductive hypothesis, we now conclude that $G_0 \circ \cdots \circ G_i$ is a homeomorphism over $N - e(\omega(e))$.

Next we show that H is injective over $N - e(\omega(e))$. To this end let x_1 and x_2 be distinct points of $H^{-1}(N - e(\omega(e)))$. Choose $i \geq 1$ so that $\rho(x_1, x_2) > 2^{-i+1}$. Then

$$2^{-i+1} < \rho(x_1, x_2) \leq \rho(x_1, H_{i+1}(x_i)) + \rho(H_{i+1}(x_1), H_{i+1}(x_2)) + \rho(H_{i+1}(x_2), x_2)$$

$$\leq 2^{-i} + \rho(H_{i+1}(x_1), H_{i+1}(x_2)) + 2^{-i} = \rho(H_{i+1}(x_1), H_{i+1}(x_2)) + 2^{-i+1}.$$

Hence, $\rho(H_{i+1}(x_1), H_{i+1}(x_2)) > 0$. So $H_{i+1}(x_1) \neq H_{i+1}(x_2)$. For $j = 1, 2$, since

$$G_0 \circ \cdots \circ G_i \circ H_{i+1}(x_j) = H(x_j) \in N - e(\omega(e)),$$

then

$$H_{i+1}(x_j) \in (G_0 \circ \cdots \circ G_i)^{-1}(N - e(\omega(e))).$$

Since $G_0 \circ \cdots \circ G_i$ is a homeomorphism over $(G_0 \circ \cdots \circ G_i)^{-1}(N - e(\omega(e)))$, we conclude that $G_0 \circ \cdots \circ G_i \circ H_{i+1}(x_1) \neq G_0 \circ \cdots \circ G_i \circ H_{i+1}(x_2)$. Therefore, $H(x_1) \neq H(x_2)$. We have established that H is injective over $N - e(\omega(e))$.

Since H is a proper map, so is $H|H^{-1}(N - e(\omega(e)))$. Thus, over $N - e(\omega(e))$, H is a closed injective map and, hence, a homeomorphism. $\square$

6. Resolving wild embeddings of a generalized $(n-1)$-manifold in an n-manifold

With only simple modifications, the preceding proof generalizes to the case of an embedding $e: M \to N$ of a *generalized* $(n-1)$-manifold M in an n-manifold N. We describe these modifications in this section.

Let $e: M \to N$ be an embedding of a generalized $(n-1)$-manifold in an n-manifold. The definitions of e being *tame*, the *tame set* $\tau(e)$ of e, the *wild set* $\omega(e)$ of e, and a δ-*resolution* of e are verbatim the same as those given in Section 2.

Before stating the appropriate resolution theorem for wild embeddings of generalized $(n-1)$-manifolds in n-manifolds, we make a relevant observation. If an embedding of a generalized $(n-1)$-manifold M in an n-manifold N is tame, then $M \times \mathbb{R}$ is an n-manifold. The reason is that M is covered by open subsets U such that $U \times \mathbb{R}$ embeds in N. Since each $U \times \mathbb{R}$ is a generalized n-manifold, and since generalized n-manifolds obey *invariance of domain* [1, Theorem VI.10], then each $U \times \mathbb{R}$ embeds as an open subset of N. Thus, each $U \times \mathbb{R}$ is an n-manifold. So $M \times \mathbb{R}$ is an n-manifold. Consequently, in the following theorem, the hypothesis that $M \times \mathbb{R}$ be an n-manifold is no real restriction.

Theorem 6.1 (Resolution Theorem for Wild Embeddings of Generalized $(n-1)$-Manifolds in n-Manifolds). *Suppose $e : M \to N$ is an embedding of a generalized $(n-1)$-manifold M in an n-manifold N, where $n \geq 4$, and suppose $M \times \mathbb{R}$ is an n-manifold. Let ρ be a metric on N. Then for every map $\delta : N \to [0, \infty)$ which is strictly positive on $e(\omega(e))$, there is a δ-resolution of e.*

As in Section 2, Theorem 6.1 has the following tame approximation theorem as an immediate consequence.

Theorem 6.2 (Tame Approximation Theorem for Embeddings of Generalized $(n-1)$-Manifolds in n-Manifolds). *Suppose $e : M \to N$ is an embedding of a generalized $(n-1)$-manifold M in an n-manifold N, where $n \geq 4$, and suppose $M \times \mathbb{R}$ is an n-manifold. Let ρ be a metric on N. Then for every map $\delta : M \to [0, \infty)$ which is strictly positive on $\omega(e)$, there is a tame embedding $f : M \to N$ such that $\rho(e(x), f(x)) \leq \delta(x)$ for each $x \in M$.*

The proof of Theorem 2.2 applies here without change.

To obtain a proof of Theorem 6.1, one can quote verbatim the proof of Theorem 2.1 given in Sections 4 and 5, *except* at one point. This point occurs in Section 4. We define the space Y as in Section 4. Again, Y is a generalized n-manifold for the reasons given in Section 4. However, Theorem 2.3, which was invoked in Section 4 to obtain the resolution $g : P \to Y$, is inadequate here. The reason is that in the present case, the nonmanifold set of Y lies in the set $M \times \{0, 1\}$; and $M \times \{0, 1\}$ is a *generalized* $(n-1)$-manifold, but not necessarily an $(n-1)$-manifold. We overcome this obstacle by appealing to a slightly stronger resolution theorem, stated immediately below, which covers the present situation. This is the only alteration needed to make the proof given in Sections 4 and 5 work here.

Theorem 6.3 (A Second Resolution Theorem for Certain Generalized Manifolds). *For $n \geq 4$, a generalized n-manifold Y has a resolution if the nonmanifold set of Y is contained in a closed subset X of Y such that X is a generalized $(n-1)$-manifold which has a resolution.*

Proof. $Y \times \mathbb{R}^2$ is a generalized $(n+2)$-manifold whose nonmanifold set is contained in the closed subset $X \times \mathbb{R}^2$. Since X has a resolution, so does $X \times \mathbb{R}^2$. As $\dim(X \times \mathbb{R}^2) = n+1 \geqslant 5$, it follows from [11, Corollary 2.13] and [13] that $X \times \mathbb{R}^2$ is an $(n+1)$-manifold. Now, since $n+2 \geqslant 5$, [8] provides a resolution of $Y \times \mathbb{R}^2$. Finally, two applications of Theorem 3.1 yield a resolution of Y. $\square$

We wish to apply this resolution theorem to the generalized n-manifold Y mentioned earlier in this section. The nonmanifold set of Y is contained in the generalized $(n-1)$-manifold $M \times \{0, 1\}$. Thus, it suffices to show that $M \times \{0, 1\}$ has a resolution. We are given that $M \times \mathbb{R}$ is an n-manifold; so $\mathrm{id}_{M \times \mathbb{R}}$ is a resolution of $M \times \mathbb{R}$. Since $n \geqslant 4$, Theorem 3.1 implies that M has a resolution. Hence, $M \times \{0, 1\}$ has a resolution.

We conclude this section by showing how the preceding resolution theorem can be used in conjunction with the Cell-like Approximation Theorem and the Controlled h-Cobordism Theorem to give a quick proof of the following result.

Theorem 6.4 (Resolution Uniqueness Theorem). *Suppose that $f: M \to Y$ and $g: N \to Y$ are cell-like maps from n-manifolds M and N to a generalized n-manifold Y, where $n \geqslant 4$. Let ρ be a metric on Y. Then for every map $\delta: Y \to (0, \infty)$, there is a homeomorphism $h: M \to N$ such that $\rho(f(x), g \circ h(x)) < \delta \circ f(x)$ for every $x \in M$.*

This theorem was originally proved for $n \geqslant 5$ in [18, Proposition 3.2.3]. Also see the proof of [19, Theorem 2.6.1].

Observe that the Cell-like Approximation Theorem follows from this theorem simply by setting $Y = N$ and $g = \mathrm{id}_N$.

Proof. Let $\pi_f: Z(f) \to Y$ and $\pi_g: Z(g) \to Y$ denote the usual mapping cylinder retractions. Consider the *double mapping cylinder* $Q = Z(f) \cup_Y Z(g)$, and define $\pi: Q \to Y$ by $\pi = \pi_f \cup \pi_g$.

Set $Q_0 = Q - (M \cup N)$. We argue that Q_0 is a generalized $(n+1)$-manifold. First note that $(Z(f) - M) \cup_{Y = Y \times \{1\}} (Y \times [1, 2))$ is a generalized $(n+1)$-manifold because it is the cell-like image of the $(n+1)$-manifold $M \times (0, 2)$. Then note that $Z(f) - M$ is the closure of a component of $[(Z(f) - M) \cup_{Y = Y \times \{1\}} (Y \times [1, 2))] - Y$. At this point, we conclude that $Z(f) - M$ is a generalized $(n+1)$-manifold with boundary equal to Y, by invoking Lemma 4.1. Similarly, $Z(g) - N$ is a generalized $(n+1)$-manifold with boundary equal to Y. Since $Q_0 = (Z(f) - M) \cup_Y (Z(g) - N)$, then Lemma 4.1 implies that Q_0 is a generalized $(n+1)$-manifold.

The nonmanifold set of Q_0 lies in the resolvable generalized n-manifold Y. Hence, Theorem 6.3 together with Corollary 2.6 provide a conservative resolution $F_0: P_0 \to Q_0$. F_0 extends to a cell-like map $F: P \to Q$ where P is an $(n+1)$-manifold with boundary such that $\mathrm{int}(P) = P_0$ and ∂P can be identified with the disjoint union of M and N in such a way that $F|\partial P = \mathrm{id}_{M \cup N}$.

M and N are controlled strong deformation retracts of Q with respect to the map $\pi: Q \to Y$. Indeed, Lemma 3.2 provides controlled strong deformation retractions of $Z(f)$ and $Z(g)$ onto their ends. By *stacking* these deformations, one obtains controlled strong deformation retractions of Q onto M and onto N.

The cell-like map $F: P \to Q$ is a fine homotopy equivalence. One can use the controlled homotopy inverses of F to *lift* the controlled strong deformation retractions of Q onto M and N. This yields controlled strong deformation retractions of P onto M and N. Thus, P is a controlled h-cobordism with respect to the control map $\pi \circ F: P \to Y$.

Given a map $\delta: Y \to (0, \infty)$, the Controlled h-Cobordism Theorem of [19] provides a homeomorphism $H: M \times [0, 1] \to P$ such that $H(x, 0) = x$ and $\rho(\pi \circ F(x), \pi \circ F \circ H(x, t)) < \delta \circ \pi \circ F(x)$ for each $(x, t) \in M \times [0, 1]$. Define the homeomorphism $h: M \to N$ by $h(x) = H(x, 1)$ for $x \in M$. Then $\rho(f(x), g \circ h(x)) < \delta \circ f(x)$ for every $x \in M$. $\square$

7. Resolving wild embeddings of a generalized n-manifold with boundary in an n-manifold

Recall from Section 4 that a space X is a *generalized n-manifold with boundary* if X is an ENR which has a closed subset, denoted ∂X, such that ∂X is a generalized $(n-1)$-manifold, $\text{int}(X) = X - \partial X$ is a generalized n-manifold, and $H_*(X, X - \{x\}) = 0$ for each $x \in \partial X$. ∂X is called the *boundary* of X, and $\text{int}(X)$ is called the *interior* of X.

In this section, we shall consider only those generalized n-manifolds with boundary that embed in n-manifolds. If a generalized n-manifold with boundary X embeds in an n-manifold N, then $\text{int}(X)$ must be an n-manifold. For since generalized n-manifolds obey *invariance of domain* [1, Theorem VI.10], then $\text{int}(X)$ must embed as an open subset of N. So the generalized n-manifolds with boundary arising here all have manifold interior.

Suppose X is a generalized n-manifold with boundary. Notice that even if ∂X and $\text{int}(X)$ are manifolds, X need not be an n-manifold with boundary. This occurs precisely if ∂X is not collared in X. The simplest instance of this phenomenon is a crumpled n-cube. A *crumpled n-cube* is a compact generalized n-manifold with boundary which embeds in the n-sphere and whose boundary is an $(n-1)$-sphere. To obtain a crumpled n-cube which is not an n-manifold with boundary, one takes the closure of a *bad* complementary domain of a wildly embedded $(n-1)$-sphere in an n-sphere.

Suppose $e: X \to N$ is an embedding of a generalized n-manifold with boundary X into an n-manifold N. A point x of ∂X is a *tame point* of e if there is an open neighborhood U of x in ∂X and an embedding $E_U: U \times [0, \infty) \to N$ such that $e(u) = E_U(u, 0)$ for every $u \in U$ and $e(X) \cap E_U(U \times (0, \infty)) = \varnothing$. The set of tame points of e is called the *tame set* of e and is denoted $\tau(e)$. Clearly $\tau(e)$ is an open

subset of ∂X. The set $\partial X - \tau(e)$ is called the *wild set* of e and is denoted $\omega(e)$. Thus, $\omega(e)$ is a closed subset of ∂X.

Suppose $e: X \to N$ is an embedding of a generalized n-manifold with boundary X into an n-manifold N. e is a *tame* embedding if every point of ∂X is a tame point of e. Observe that if e is tame, then $\partial X \times \mathbb{R}$ is an n-manifold. The reason is that ∂X is covered by open subsets U such that $U \times (0, \infty)$ embeds in N. Since $U \times (0, \infty)$ is a generalized n-manifold, *invariance of domain* [1, Theorem VI.10] implies that $U \times (0, \infty)$ embeds as an open subset of N. Thus, each $U \times (0, \infty)$ is an n-manifold. So $\partial X \times (0, \infty)$ is an n-manifold. The theorems stated below produce tame embeddings of X in N; consequently, in these theorems, the hypothesis that $\partial X \times \mathbb{R}$ be an n-manifold is no real restriction.

Suppose $e: X \to N$ is an embedding of a generalized n-manifold with boundary X in an n-manifold N, and suppose $\partial X \times \mathbb{R}$ is an n-manifold. We establish the following notation:

$$X^+ = X \cup_{(\partial X = \partial X \times \{0\})} \partial X \times [0, 1]$$

From Lemma 4.1, one deduces that X^+ is a generalized n-manifold with boundary equal to $\partial X \times \{1\}$. So

$$\text{int}(X^+) = X \cup_{(\partial X = \partial X \times \{0\})} \partial X \times [0, 1)$$

is a generalized n-manifold. Observe that if $f^+: X^+ \to N$ is an embedding, then $f^+|X: X \to N$ is a tame embedding.

Suppose $e: X \to N$ is an embedding of a generalized n-manifold with boundary X into an n-manifold N such that $\partial X \times \mathbb{R}$ is an n-manifold, and suppose ρ is a metric on N. Let $\delta: N \to [0, \infty)$ be a map. A *δ-resolution* of e is a cell-like map $G: N \to N$ to which is associated a tame embedding $f: X \to N$ such that $G \circ f = e$, G is a homeomorphism over $N - e(\omega(e))$, and $\rho(x, G(x)) \leq \delta(x)$ for every $x \in N$. We say that this δ-resolution is *collared* if the tame embedding $f: X \to N$ extends to an embedding $f^+: X^+ \to N$ such that $G^{-1}(e(x)) = f^+(\{x\} \times [0, 1])$ for each $x \in \omega(e)$. Thus, when G is collared, we have more precise information about the point inverses of G: each point inverse of G is either a point or an arc fiber of $f^+(\partial X \times [0, 1])$ over a point of $\omega(e)$. Collared δ-resolutions are the sort of δ-resolutions which arise naturally when considering embeddings of generalized n-manifolds with boundary in n-manifolds.

The versions of the Resolution Theorem for Wild Codimension-One Embeddings and the Codimension-One Tame Approximation Theorem occurring in previous sections can be reformulated for embeddings of generalized n-manifolds with boundary in n-manifolds. This leads to the following three conjectures.

Conjecture 7.1. *Suppose $e: X \to N$ is an embedding of a generalized n-manifold with boundary X in an n-manifold N, where $n \geq 4$, and suppose $\partial X \times \mathbb{R}$ is an n-manifold. Let ρ be a metric on N. Then for every map $\delta: N \to [0, \infty)$ which is strictly positive on $e(\omega(e))$, there is a collared δ-resolution of e.*

Conjecture 7.2. *Suppose $e : X \to N$ is an embedding of a generalized n-manifold with boundary X in an n-manifold N, where $n \geq 4$, and suppose $\partial X \times \mathbb{R}$ is an n-manifold. Let ρ be a metric on N. Then for every map $\delta : X \to [0, \infty)$ which is strictly positive on $\omega(e)$, there is a tame embedding $f : X \to N$ such that $\rho(e(x), f(x)) \leq \delta(x)$ for each $x \in X$.*

Conjecture 7.3. *Suppose X is a generalized n-manifold with boundary, where $n \geq 4$, such that X embeds in an n-manifold and $\partial X \times \mathbb{R}$ is an n-manifold. Then $\mathrm{int}(X^+)$ is an n-manifold.*

In this section, we shall prove that these three conjectures are equivalent, and that they are true in dimensions $n \geq 5$. We shall also describe the results of Daverman [12] which establish a special case of these conjectures in dimension $n = 4$.

Theorem 7.4. *Conjectures 7.1, 7.2 and 7.3 are equivalent.*

Proof. A proof that Conjecture 7.1 implies Conjecture 7.2 can be adapted from the proof in Section 2 that Theorem 2.1 implies Theorem 2.2. Simply change M to X.

Assume Conjecture 7.2. Begin by observing that Lemma 4.1 implies that $\mathrm{int}(X^+)$ is a generalized n-manifold. Conjecture 7.2 provides a tame embedding $f : X \to N$ of X in an n-manifold N. From [7], we conclude that f extends to an embedding $f^+ : X^+ \to N$. Now, *invariance of domain* [1, Theorem VI.10] implies that $f^+(\mathrm{int}(X^+))$ is an open subset of N. So $\mathrm{int}(X^+)$ is an n-manifold.

The proof that Conjecture 7.3 implies Conjecture 7.1 is a simplified version of the proof in Section 4. The simplification results from the fact that Conjecture 7.3 implies that the space Y, defined in Section 4, is actually an n-manifold.

Assume Conjecture 7.3. Suppose $e : X \to N$ is an embedding of a generalized n-manifold with boundary X in an n-manifold N, where $n \geq 4$, and suppose $\partial X \times \mathbb{R}$ is an n-manifold.

Since $e(X)$ is locally compact, there is an open subset N_0 of N such that $e(X)$ is a closed subset of N_0. Brown [7] provides a *collar* on $e(\tau(e))$ in $N_0 - e(\mathrm{int}(X))$. By restricting this collar and tapering it near $e(\omega(e))$, we obtain a closed map $c : (\partial X \times [0, 1]) \to (N_0 - e(\mathrm{int}(X)))$ such that $c(x, 0) = e(x)$ for each $x \in \partial X$, $c|\tau(e) \times [0, 1]$ is an embedding, and $c^{-1}(e(x)) = \{x\} \times [0, 1]$ for every $x \in \omega(e)$.

Set

$$Z = \mathrm{cl}_{N_0}[N_0 - (e(X) \cup c(\partial X \times [0, 1]))]$$

$$= [N_0 - (e(X) \cup c(\partial X \times [0, 1]))] \cup c(\partial X \times \{1\}).$$

Lemma 4.1 implies that Z is a generalized n-manifold with boundary equal to $c(\partial X \times \{1\})$. Also, $Z \subset N$, and $\partial Z \times \mathbb{R}$ is an n-manifold because ∂Z is a homeomorphic to ∂X. Hence, Conjecture 3 implies that $\mathrm{int}(X^+)$ and $\mathrm{int}(Z^+)$ are n-manifolds.

Set

$$Y = X_{(c|\partial X \times \{0\})} \cup \partial X \times [0, 1] \cup_{(c|\partial X \times \{1\})} Z.$$

In other words, to obtain Y, remove $c(\partial X \times [0, 1])$ from N and sew in $\partial X \times [0, 1]$,

using $c|\partial X \times \{0\}$ to attach $\partial X \times \{0\}$ to ∂X, and using $c|\partial X \times \{1\}$ to attach $\partial X \times \{1\}$ to ∂Z. We identify X^+ and Z^+ with the two subsets

$$X_{(c|\partial X \times \{0\})} \cup \partial X \times [0, 1] \quad \text{and} \quad \partial X \times [0, 1] \cup_{(c|\partial X \times \{1\})} Z$$

of Y. This identifies $\text{int}(X^+)$ and $\text{int}(Z^+)$ with the two open subsets

$$X_{(c|\partial X \times \{0\})} \cup \partial X \times [0, 1) \quad \text{and} \quad \partial X \times (0, 1] \cup_{(c|\partial X \times \{1\})} Z$$

of Y. Since the union of these two open sets is Y, and since $\text{int}(X^+)$ and $\text{int}(Z^+)$ are n-manifolds, then Y is an n-manifold.

Define the map $g : Y \to N_0$ by $g(x, t) = c(x, t)$ for each $(x, t) \in \partial X \times [0, 1]$, $g|X = e$ and $g|Z = \text{id}_Z$. Then $g^{-1}(e(x)) = \{x\} \times [0, 1]$ if $x \in \omega(e)$, and $g^{-1}(x)$ is a point if $x \in N_0 - e(\omega(e))$. Therefore, g is a cell-like map and a homeomorphism over $N_0 - e(\omega(e))$.

Suppose ρ is a metric on N, and $\delta : N \to [0, \infty)$ is a map which is strictly positive on $e(\omega(e))$. There is a map $\delta_0 : N \to [0, \infty)$ such that $\delta_0 \leq \delta$, δ_0 is strictly positive on $e(\omega(e))$, and $\delta_0 = 0$ on $N - N_0$. Next, there is a map $\gamma : N \to [0, \infty)$ such that γ is strictly positive on $e(\omega(e))$, $\gamma = 0$ on $N - N_0$, and γ has the following property: if $\rho(x, y) \leq \gamma(y)$, then $\rho(x, y) \leq \delta_0(x)$ for all x and y in N. Set $U = \gamma^{-1}(0, \infty)$. Then U is an open subset of N_0 which contains $e(\omega(e))$. Hence, g is a homeomorphism over $N_0 - U$.

Theorem 2.4 provides a homeomorphism $h_U : g^{-1}(U) \to U$ such that $\rho(g(x), h_U(x)) < \gamma \circ g(x)$ for every $x \in g^{-1}(U)$. Define the function $h : Y \to N_0$ by

$$h = h_U \cup g|g^{-1}(N_0 - U).$$

Since $\gamma = 0$ on $N_0 - U$ and g is a homeomorphism over $N_0 - U$, then h is a homeomorphism. Clearly, $\rho(g(x), h(x)) \leq \gamma \circ g(x)$ for every $x \in Y$.

Define the function $G : N \to N$ by setting $G = g \circ h^{-1} \cup \text{id}_{N - N_0}$. Note that for each $x \in N_0$, since $\rho(x, g \circ h^{-1}(x)) = \rho(h \circ h^{-1}(x), g \circ h^{-1}(x)) \leq \gamma \circ g \circ h^{-1}(x)$, then $\rho(x, g \circ h^{-1}(x)) \leq \delta_0(x)$. Hence, $\rho(x, G(x)) \leq \delta_0(x) \leq \delta(x)$ for each $x \in N$. As $\delta_0 = 0$ on $N - N_0$, we conclude that G is continuous.

$G : N \to N$ is a cell-like map because the map $g \circ h^{-1} : N_0 \to N_0$ is cell-like. Since g is a homeomorphism over $N_0 - e(\omega(e))$, so is $g \circ h^{-1}$; consequently G is a homeomorphism over $N - e(\omega(e))$. On the other hand, if $x \in \omega(e)$, then $G^{-1}(e(x)) = h(g^{-1}(e(x))) = h(\{x\} \times [0, 1])$.

Define the embeddings $f : X \to N$ and $f^+ : X^+ \to N$ by $f = h|X$ and $f^+ = h|X^+$. Then f is tame, f^+ is an extension of f, and $G \circ f = (g \circ h^{-1}) \circ (h|X) = g|X = e$. Also, if $x \in \omega(e)$, then $G^{-1}(e(x)) = h(\{x\} \times [0, 1]) = f^+(\{x\} \times [0, 1])$. We conclude that G is a collared δ-resolution of e. $\square$

Theorem 7.5. *Conjectures 7.1, 7.2 and 7.3 are true in dimensions $n \geq 5$.*

Proof. Since the three conjectures are equivalent, it suffices to verify Conjecture 7.3.

Suppose $n \geq 5$, X is a generalized n-manifold with boundary that embeds in an n-manifold N, and $\partial X \times \mathbb{R}$ is an n-manifold. We regard X as a subset of N. We

can assume that X is a closed subset of N. This is because there is an open subset N_0 of N such that X is a closed subset of N_0, and we can replace N by N_0 if necessary.

Let ρ be a metric on N. We define a metric σ on X^+ by the following formulas.

$$\sigma(x, y) = \rho(x, y) \quad \text{for } x \text{ and } y \text{ in } X$$

$$\sigma((x, s), (y, t)) = \rho(x, y) + |s - t| \quad \text{for } (x, s) \text{ and } (y, t) \text{ in } \partial X \times [0, 1]$$

$$\sigma(x, (y, t)) = \rho(x, y) + |t| \quad \text{for } x \in X \text{ and } (y, t) \in \partial X \times [0, 1]$$

$\text{int}(X^+)$ is a generalized n-manifold. Moreover its nonmanifold set lies in $\partial X = \partial X \times \{0\}$, because $\text{int}(X)$ and $\partial X \times (0, 1)$ are n-manifolds. Since ∂X is a generalized $(n-1)$-manifold and a closed subset of $\text{int}(X^+)$, then Theorem 6.3 provides a resolution of $\text{int}(X^+)$. Therefore, according to [13], in order to prove that $\text{int}(X^+)$ is an n-manifold, it suffices to show that $\text{int}(X^+)$ has the *disjoint disks property*. To this end, suppose $f_1 : D \to \text{int}(X^+)$ and $f_2 : D \to \text{int}(X^+)$ are maps of a two-dimensional disk D into $\text{int}(X^+)$, and let $\varepsilon > 0$. We must find maps $k_1 : D \to \text{int}(X^+)$ and $k_2 : D \to \text{int}(X^+)$ such that $k_1(D) \cap k_2(D) = \varnothing$ and $\sigma(f_i(x), k_i(x)) < \varepsilon$ for each $x \in D$, for $i = 1, 2$.

We shall need the following auxiliary maps. Define the map $\pi : X^+ \to X$ by $\pi | X = \text{id}_X$ and $\pi(x, t) = x$ for $(x, t) \in \partial X \times [0, 1]$. Define the map $\lambda : X^+ \to [0, 1]$ by $\lambda(X) = 0$ and $\lambda(x, t) = t$ for $(x, t) \in \partial X \times [0, 1]$. Since ∂X is an absolute neighborhood retract, there is an open neighborhood U of ∂X in N and a map $r : U \to \partial X$ such that $r | \partial X = \text{id}_{\partial X}$ and $\rho(x, r(x)) < \frac{1}{4}\varepsilon$ for each $x \in U$.

Step 1. Set $A_i = f_i^{-1}(X)$ and $B_i = f_i^{-1}(\partial X \times [\frac{1}{4}\varepsilon, 1))$. A_i and B_i are disjoint closed subsets of D. Hence, there is a compact 2-manifold with boundary E_i in D such that $A_i \subset \text{int}(E_i)$ and $E_i \cap B_i = \varnothing$. For $i = 1, 2$, we shall construct maps $g_i : D \to (X \cup U)$ such that $g_i(\partial E_i) \subset U - X$, $g_i(D - E_i) \subset U$, $\rho(\pi \circ f_i(x), g_i(x)) < \frac{1}{4}\varepsilon$ for each $x \in D$, and $g_1(D) \cap g_2(D) = \varnothing$.

The construction of g_i relies on the fact that the inclusion of ∂X in U is *locally homologically 0 co-connected* in $U - X$ [1, Theorem VI.6]. This means that nearby points in $U - X$ can be joined by small arcs in $U - X$. We use this fact to approximate $\pi \circ f_i | \partial E_i : \partial E_i \to \partial X$ by a map $\gamma_i : \partial E_i \to U - X$. To obtain γ_i, we triangulate ∂E_i very finely, and we let γ_i map each vertex v of ∂E_i into $U - X$ very near $\pi \circ f_i(v)$. We then invoke the local homological 0 co-connectivity in $U - X$ to define γ_i on each edge e of ∂E_i so that $\gamma_i(e)$ has very small diameter. This results in a map $\gamma_i : \partial E_i \to (U - X)$ which is very close to $\pi \circ f_i | \partial E_i : \partial E_i \to \partial X$.

Since U is an absolute neighborhood retract, there is a homotopy between $\pi \circ f_i | \partial E_i$ and γ_i in U of track diameter $< \frac{1}{4}\varepsilon$ (assuming that γ_i is sufficiently close to $\pi \circ f_i | \partial E_i$). The Controlled Borsuk Homotopy Extension Principle (Theorem 3.4) extends this homotopy to a homotopy in U of track diameter $< \frac{1}{4}\varepsilon$ from the map $\pi \circ f_i | \text{cl}(D - A_i) : \text{cl}(D - A_i) \to \partial X$ to a map $\Gamma_i : \text{cl}(D - A_i) \to U$ such that $\Gamma_i = \gamma_i$ on ∂E_i and $\Gamma_i = \pi \circ f_i$ on the frontier of A_i. Then, $\Gamma_i \cup \pi \circ f_i | A_i : D \to (X \cup U)$ is a map that sends ∂E_i into $U - X$, sends $D - E_i$ into U, and is within $\frac{1}{4}\varepsilon$ of $\pi \circ f_i$.

Since $\dim(N) = n \geqslant 5$, then slight general position perturbations of the maps $\Gamma_i \cup \pi \circ f_i | A_i$ ($i = 1, 2$) will produce maps $g_i : D \to (X \cup U)$ with disjoint images as well as the other desired properties.

Step 2. Let $C_i = g_i^{-1}(X) \cap E_i$. C_i is a closed subset of D. Moreover, $C_i \subset \text{int}(E_i)$ because $g_i(\partial E_i) \subset U - X$. Define the map $s : (X \cup U) \to X$ by $s = \text{id}_X \cup r|\text{cl}(U - X)$. Since g_i maps E_i into $X \cup U$, ∂E_i into $U - X$, and $D - E_i$ into U, then a map $h_i : D \to X$ is defined by the formula $h_i = (s \circ g_i | E_i) \cup (r \circ g_i | D - \text{int}(E_i))$. Then $h_i | C_i = g_i | C_1$, $h_i(\text{cl}(D - C_i)) \subset \partial X$, and $\rho(g_i(x), h_1(x)) < \frac{1}{4}\varepsilon$ for each $x \in D$.

Step 3. Next with the aid of the Tietze Extension Theorem, we obtain a map $\varphi_i : D \to [0, 1)$ such that $\varphi_i(C_i) = 0$, $\varphi_i | B_i = \lambda \circ f_i | B_i$, and $\varphi_i(D - (C_i \cup B_i)) \subset (0, \frac{1}{4}\varepsilon)$. Then $\varphi_i(D - B_i) \subset [0, \frac{1}{4}\varepsilon)$ and $|\varphi_i(x) - \lambda \circ f_i(x)| < \frac{1}{4}\varepsilon$ for every $x \in D$. Define the map $j_i : D \to \text{int}(X^+)$ by setting $j_i | C_i = h_i | C_i$ and letting $j_i(x) = (h_i(x), \varphi_i(x))$ for each $x \in \text{cl}(D - C_i)$. Then $j_i | C_i = g_i | C_i : C_i \to X$ and $j_i(D - C_i) \subset \partial X \times (0, 1)$.

We assert that $\sigma(f_i(x), j_i(x)) < \varepsilon$ for every $x \in D$. First, we consider the case in which $x \in D - B_i$. Then

$$\sigma(f_i(x), j_i(x)) \leqslant \sigma(f_i(x), \pi \circ f_i(x)) + \sigma(\pi \circ f_i(x), h_i(x)) + \sigma(h_i(x), j_i(x)).$$

Observe that $\sigma(f_i(x), \pi \circ f_i(x)) = \lambda \circ f_i(x) < \frac{1}{4}\varepsilon$ because $x \in D - B_i$,

$$\sigma(\pi \circ f_i(x), h_i(x)) = \rho(\pi \circ f_i(x), h_i(x))$$
$$\leqslant \rho(\pi \circ f_i(x), g_i(x)) + \rho(g_i(x), h_i(x)) < \frac{1}{2}\varepsilon,$$

and $\sigma(h_i(x), j_i(x)) = \varphi_i(x) < \frac{1}{4}\varepsilon$ because $x \in D - B_i$. Hence, $\sigma(f_i(x), j_i(x)) < \varepsilon$. Second, suppose that $x \in B_i$. Then $f_i(x) = (\pi \circ f_i(x), \lambda \circ f_i(x))$, and $j_i(x) = (h_i(x), \varphi_i(x)) = (h_i(x), \lambda \circ f_i(x))$. Hence,

$$\sigma(f_i(x), j_i(x)) = \rho(\pi \circ f_i(x), h_i(x))$$
$$\leqslant \rho(\pi \circ f_i(x), g_i(x)) + \rho(g_i(x), h_i(x)) < \frac{1}{2}\varepsilon < \varepsilon.$$

This proves the assertion.

Step 4. Observe that $j_i(C_i) = g_i(C_i) \subset X$, $j_i(D - C_i) \subset \partial X \times (0, 1)$, and $j_1(C_1) \cap j_2(C_2) = g_1(C_1) \cap g_2(C_2) = \varnothing$. Since $\partial X \times (0, 1)$ is a manifold of dimension $\geqslant 5$, we can perform a slight general position perturbation on $j_i | D - C_i$, damping out this perturbation as we near $j_i | C_i$, to obtain maps $k_i : D \to \text{int}(X^+)$ such that $k_1(D - C_1) \cap k_2(D - C_2) = \varnothing$, $k_i(D - C_i) \subset \partial X \times (0, 1)$, $k_i | C_i = j_i | C_i$, and $\sigma(f_i(x), k_i(x)) < \varepsilon$ for every $x \in D$. It follows that $k_1(D) \cap k_2(D) = \varnothing$. $\square$

We shall say that a generalized n-manifold with boundary X is ∂-*nice* if X embeds in an n-manifold and ∂X is an $(n - 1)$-manifold (not merely a generalized $(n - 1)$-manifold). In dimension $n = 4$, Conjectures 7.1–7.3 in their full generality have not yet been resolved. However, Daverman has proved these conjectures in dimension $n = 4$ (as well as in dimensions $n \geqslant 5$) for the class of embeddings of all ∂-nice generalized n-manifolds with boundary in n-manifolds. We shall review the outline of Daverman's work. However, we shall first discuss the equivalence of Conjectures 7.1–7.3 when restricted to special classes of embeddings such as the one just mentioned.

Conjectures 7.1–7.3 are as strong as possible in the sense that each applies to the class of all embeddings of generalized n-manifolds with boundary in n-manifolds that could conceivably have tame approximations. Specifically, these three conjectures apply to:

(1) The class of all embeddings of generalized n-manifolds with boundary X in n-manifolds such that $\partial X \times \mathbb{R}$ is an n-manifold.

We proved above that the three conjectures (as they apply to this widest possible class of embeddings) are equivalent. We assert that the proof of equivalence given above remains valid if the three conjectures are restricted to either of the two following narrower classes of embeddings.

(2) The class of all embeddings of ∂-nice generalized n-manifolds with boundary in n-manifolds.

(3) The class of all embeddings of crumpled n-cubes in the n-sphere.

In checking that the proof of the equivalence of Conjectures 7.1–7.3 adapts to the case in which the conjectures have been restricted one of the special classes of embeddings (2) or (3), there is only one delicate point. This occurs in the proof that Conjecture 7.3 implies Conjecture 7.1, where Conjecture 7.3 must be applied to the inclusion $Z \subset N$ as well as to the original embedding $e : X \to N$. For the argument to be valid, it must be shown that if the embedding $e : X \to N$ belongs to the class under consideration ((2) or (3)), then so does the inclusion $Z \subset N$. In the case of class (2), we are given that ∂X is an $(n-1)$-manifold, and we must establish that ∂Z is an $(n-1)$-manifold; but this is immediate because ∂Z is homeomorphic to ∂X. In the case of class (3), we are given that X is a crumpled n-cube and $N = S^n$, and we must establish that Z is a crumpled n-cube; but this follows easily because Z is a closed and, hence, compact subset of $N = S^n$ and ∂Z is homeomorphic to the $(n-1)$-sphere ∂X. This proves our assertion that Conjectures 7.1–7.3 remain equivalent when restricted to either of the special classes of embeddings (2) or (3).

We now resume our discussion of Daverman's results. Daverman first established Conjectures 7.1–7.3 in dimensions ≥ 4 for the class (3) of all embeddings of crumpled n-cubes in the n-sphere. Specifically, in [9], he proved Conjecture 2 for embeddings of crumpled n-cubes in the n-sphere in dimensions $n \geq 5$, using the Resolution Theorem for Wild Codimension-One Embeddings of $(n-1)$-manifolds in n-manifolds which at that time had just been established in dimensions $n \geq 5$ in [3]. In the recent preprint [12], Daverman proved Conjecture 7.2 for embeddings of crumpled 4-cubes in the 4-sphere, using the Resolution Theorem for Wild Codimension-One Embeddings of 3-manifolds in 4-manifolds which appears for the first time in this paper. Since Conjectures 7.1–7.3 are equivalent for embeddings of crumpled n-cubes in the n-sphere, then Daverman's work establishes Conjecture 7.3 in dimensions ≥ 4 for all crumpled cubes.

Daverman found a clever argument (the proof of [10, Theorem 5B.10]) which enabled him to promote his proof of Conjectures 7.1–7.3 from the class (3) of crumpled cubes to the class (2) of all ∂-nice generalized n-manifolds with boundary. Here is the outline. Suppose X is a ∂-nice generalized n-manifold with boundary.

We shall indicate why $\text{int}(X^+)$ is an n-manifold, thereby verifying Conjecture 7.3. Since $\text{int}(X)$ and $\partial X \times (0, 1)$ are n-manifolds, it suffices to prove that each point of $\partial X = \partial X \times \{0\}$ has an open n-manifold neighborhood in $\text{int}(X^+)$. Let $x \in \partial X = \partial X \times \{0\}$. Daverman [10, Theorem 5B.10] implies that there is an open neighborhood U of x in X, a crumpled n-cube C, and an embedding $f: U \to C$ such that $f(U) \cap \partial C = f(\partial U)$. (Note that U is itself a generalized n-manifold with boundary such that $\partial U = U \cap \partial X$.) Clearly, f extends to a homeomorphism of $U \cup_{(\partial U = \partial U \times \{0\})} \partial U \times [0, 1)$ onto an open subset of $\text{int}(C^+)$. $\text{int}(C^+)$ is an n-manifold, since Daverman has verified Conjecture 7.3 for crumpled cubes. Hence, $U \cup_{(\partial U = \partial U \times \{0\})} \partial U \times [0, 1)$ is an open n-manifold neighborhood of x in $\text{int}(X^+)$. This establishes Conjecture 7.3 in dimensions $n \geq 4$ for class (2). Since Conjectures 7.1, 7.2 and 7.3 are equivalent for this class of embeddings, we conclude that Conjectures 7.1, 7.2 and 7.3 are all valid in dimensions $n \geq 4$ for the class (2) of all embeddings of ∂-nice generalized n-manifolds with boundary in n-manifolds.

The theorems in this section together with Daverman's results leave unresolved only the most general case of Conjectures 7.1–7.3 in dimension 4.

Problem. Prove Conjectures 7.1, 7.2 and 7.3 in dimension $n = 4$ for all embeddings of generalized 4-manifolds with boundary X in 4-manifolds, where $\partial X \times \mathbb{R}$ is a 4-manifold, but ∂X is not necessarily a 3-manifold.

Recall that Edwards in [13] established the *disjoint disks property* as the basic criterion for detecting manifolds in dimension ≥ 5. Also recall that the disjoint disks property played a crucial role in our proof of Conjectures 7.1–7.3 in dimensions ≥ 5. Informed by these observations, we speculate that a solution to the preceding problem will come with the discovery of a criterion for detecting 4-manifolds analogous to the disjoint disks property.

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