COMPLEMENTARY 1-ULC PROPERTIES FOR 3-SPHERES IN 4-SPACE

BY

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1. Introduction

A recurring theme in geometric topology is the importance of the 1-ULC property. For example, if Σ is an $(n-1)$-sphere topologically embedded in S^n ($n \neq 4$), then Σ is flat if and only if $S^n - \Sigma$ is 1-ULC. (See [2], [8], and [5].) If U is a component of $S^n - \Sigma$, it is natural to ask: For which sets $T \subset \Sigma$ is it true that $U \cup T$ is 1-ULC? (Of course, $T = \Sigma$ always works.) For $n = 3$, R. H. Bing has shown [3] that for some 0-dimensional $T \subset \Sigma$, $U \cup T$ is 1-ULC. For $n \geq 5$, Robert J. Daverman has found [6] a 1-dimensional $T \subset \Sigma$ such that $U \cup T$ is 1-ULC. (It is suspected that the dimension of Daverman's set cannot, in general, be lowered but no example is yet at hand.)

In Theorem 1 we extend Daverman's result to cover the case $n = 4$. Moreover, as constructed our 1-dimensional set T is easily seen to have embedding dimension at most 1 relative to Σ ("$\text{dem}_\Sigma T \leq 1$"), in the sense of [13] and [10]. We cannot hope to strengthen Daverman's high-dimensional result to obtain $\text{dem}_\Sigma T \leq 1$, when $n \geq 5$. For in Theorem 2 we observe that when $n \geq 6$, if T can be found with $\text{dem}_\Sigma T \leq 1$, then T can be chosen so that $\text{dem}_\Sigma T \leq 0$. But in [7], Daverman constructs embeddings of Σ in S^n, for all $n \geq 4$, for which T can never be chosen to have $\text{dem}_\Sigma T \leq 0$. In fact, in these examples T must satisfy $\text{dem}_\Sigma T \geq n - 3$.

We account for our inability to obtain $\text{dem}_\Sigma T \leq 1$ when $n > 4$ by remarking that for a σ-compactum T in Σ, "$\text{dem}_\Sigma T \leq 1$" is a stronger statement when $\dim \Sigma > 3$ than when $\dim \Sigma = 3$. For when $\dim \Sigma > 3$, $\text{dem}_\Sigma T \leq 1$ implies $\Sigma - T$ is 1-ULC. No such implication holds when $\dim \Sigma = 3$. We can appreciate the relative weakness of the statement "$\text{dem}_\Sigma T \leq 1$" when $\dim \Sigma = 3$ in another way: James W. Cannon has observed that when $\dim \Sigma = 3$, "$\text{dem}_\Sigma T \leq 1$" is equivalent to the existence of a 0-dimensional subset S of T such that $(\Sigma - T) \cup S$ is 1-ULC. However when $\dim \Sigma > 3$, any codimension 2 σ-compactum T in Σ contains a 0-dimensional subset S for which $(\Sigma - T) \cup S$ is 1-ULC.

Examples are easily constructed in all dimensions $n \geq 3$ with the property that any subset T of Σ for which $U \cup T$ is 1-ULC must be dense in Σ. Thus the subset T constructed by Daverman and the present authors is, in general, noncompact. In fact, Carl Pixley has noted that for $n \geq 5$, if T is a *compact*

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1-dimensional subset of Σ for which $U \cup T$ is 1-ULC, then three is a (possibly noncompact) 0-dimensional subset T' of T for which $U \cup T'$ is 1-ULC. A proof of Pixley's observation can be based on the last statement of the previous paragraph. The extension of this proof to the case $n = 4$ seems to require the stronger hypothesis "$\operatorname{dem}_\Sigma T \leq 1$." This suggests that for $n = 4$, "$\operatorname{dem}_\Sigma T \leq 1$" is an important part of the conclusion of Theorem 1.

Throughout the paper, Σ will denote an $(n-1)$-sphere topologically embedded in S^n. Let U be a component of $S^n - \Sigma$, and put $C = \operatorname{Cl} U$. (Cl denotes closure.) We will denote an n-simplex by Δ^n, with $\partial\Delta^{n+1} = S^n$. Suppose Y and Y' are metric spaces, $Y \subset Y'$. Let $k \geq 0$ be an integer. Then Y is k-ULC *in* Y' if for each $\varepsilon > 0$ there is a $\delta > 0$ such that each mapping (= continuous function) of $\partial\Delta^{k+1}$ into a subset of Y of diameter less than δ can be extended to a map of Δ^{k+1} into an ε-subset of Y'. (Usually, $Y = U$, $Y' = U \cup T$, and $k = 0$ or 1.) Also we would say "Y is k-ULC" rather than "Y is k-ULC in Y".

Here is a summary of some basic facts.

PROPOSITION 1. *The notation is as above.*

(a) *C is* 0-ULC *and* 1-ULC. (*In fact, C is a compact absolute retract and hence is uniformly locally contractible.*)

(b) *U is* 0-ULC.

(c) *If $f: \Delta^2 \to C$ is a map, P is a closed 1-dimensional subpolyhedron of Δ^2 and $\varepsilon > 0$, then there is a map $f': \Delta^2 \to C$ such that $f'(P) \subset U$ and $d(f, f') < \varepsilon$.*

(d) *If $T \subset \Sigma$, then $U \cup T$ is* 0-ULC.

(e) *If $T \subset \Sigma$ and U is* 1-ULC *in $U \cup T$, then $U \cup T$ is* 1-ULC.

(f) *If $T \subset \Sigma$ and $U \cup T$ is* 1-ULC, *then there is a σ-compact subset T' of T such that $U \cup T'$ is* 1-ULC.

Reference [4] is a compendium of information on ULC properties. In particular, a proof of Proposition 1 can be extracted from the statements and proofs of Propositions 2A, 2B.1, 2C.2, 2C.2.1, 2C.3, and 2C.7(2) of [4].

Suppose G is an open subset of a PL manifold Q and that there is a compact subpolyhedron X of Q of dimension at most k such that $X \subset G$, there is a compact metric space Y, and there is a continuous proper surjection $p: Y \times (0, 1] \to G$ such that $p^{-1}(X) = Y \times \{1\}$, $p \mid Y \times (0, 1)$ is injective and

$$\operatorname{diam} p(\{y\} \times (0, 1]) < \varepsilon \quad \text{for every } y \in Y.$$

In this situation, we say that G is an *open ε-mapping cylinder neighborhood of X in Q* and that X is a *k-spine of G*.

Suppose X is a nonempty compact subset of the interior of a PL manifold Q. For an integer $k \geq 0$, we say that the *dimension of the embedding of X in Q is at most k* (abbreviated $\operatorname{dem}_Q X \leq k$), if for each $\varepsilon > 0$, X is contained in an open ε-mapping cylinder neighborhood with a k-spine in Int Q. We say that the *dimension of the embedding of X in Q is k* (abbreviated $\operatorname{dem}_Q X = k$) if $\operatorname{dem}_Q X \leq k$ but not $\operatorname{dem}_Q X \leq k - 1$. For a nonempty σ-compact subset F

of the interior of a *PL* manifold Q, we define the *dimension of the embedding of* F *in* Q (abbreviated $\operatorname{dem}_Q F$) to be

$$\max \{\operatorname{dem}_Q X\colon X \text{ is a compact subset of } F\}.$$

We put $\operatorname{dem}_Q \emptyset = -\infty$.

PROPOSITION 2. *Suppose* F *is a* σ*-compact subset of the interior of a PL manifold* Q ($\dim Q = q$). *Then:*

(a) $\dim F \leq \operatorname{dem}_Q F$,

(b) *Suppose* $k \geq 0$ *is an integer. Then* $\operatorname{dem}_Q F \leq k$ *if and only if for each closed subpolyhedron* P *of* Q *of dimension at most* $q - k - 1$, *there is an ambient isotopy of* Q *which pushes* P *off* F, *is arbitrarily close to the identity on* Q *and is fixed outside an arbitrarily tight neighborhood of* $P \cap F$.

(c) *Suppose* $F = \bigcup_{i=1}^{\infty} X_i$ *where each* X_i *is compact for* $i = 1, 2, 3, \ldots$. *Then*

$$\operatorname{dem}_Q F = \max \{\operatorname{dem}_Q X_i\colon i = 1, 2, 3, \ldots\}.$$

Reference [10] is a comprehensive source about the dimension of an embedding. The proof of Proposition 2 follows from Propositions 1.1(1), 1.1(4), 1.2(2′), and 2.2(2′) of [10].

We end this section by establishing some notation and recalling a well-known method of obtaining an open ε-mapping cylinder neighborhood with a k-spine in the interior of a *PL* manifold. First suppose K is a simplicial complex: then for $i = 0, 1, 2, \ldots$, let $K^i = \{\alpha \in K\colon \dim \alpha \leq i\}$, the *i-skeleton* of K. Let $|K|$ denote the union of all the simplices of K; and let K' stand for some first derived subdivision of K. Second, suppose Q^q is a *PL* q-manifold, $\varepsilon > 0$ and k is one of the integers $0, 1, \ldots, q - 1$. Let K be a simplicial complex of mesh less than ε which triangulates Q^q. If L is a subcomplex such that

$$\partial Q^q \cup |K^{q-k-1}| \subset |L|,$$

and if we let

$$L_* = \{\alpha \in K'\colon \alpha \cap |L| = \emptyset\}$$

(the subcomplex of K' which is *dual* to L), then $L_* \subset (K')^k$, $|L_*| \cap \partial Q^q = \emptyset$, and $Q^q - |L|$ is an open ε-mapping cylinder neighborhood with a k-spine $|L_*|$ in Int Q^q.

2. Topologically planar subsets of 3-manifolds

Theorem 3 of [12] is the foundation of the results of this section. Before describing this theorem, we define a topological space to be *topologically planar* if it can be embedded in the Euclidean plane, R^2. Furthermore, let us observe that Proposition 2(b) implies that a compact subset X of Euclidean 3-space R^3 has the "strong arc pushing property" as defined in [12] *if and only if* $\operatorname{dem}_{R^3} X \leq 1$. Consequently, Theorem 3 of [12], when restricted to compacta, translates into the following proposition: *If* X *is a compact, topologically* *planar subset of* R^3 *and* $\dim X \leq 1$, *then* $\operatorname{dem}_{R^3} X \leq 1$. We extend this proposition slightly to a form which is more convenient for our purposes.

PROPOSITION 3. *If* X *is a compact, topologically planar subset of the interior of a PL 3-manifold* Q^3 *and* $\dim X \leq 1$, *then* $\operatorname{dem}_{Q^3} X \leq 1$.

Proof. Let $X = \bigcup_{i=1}^k X_i$ so that for each $i = 1, 2, \ldots, k$, X_i is a compactum, R_i is an open subset of Int Q^3 which is PL homeomorphic to R^3, and $X_i \subset R_i$. Thus $\operatorname{dem}_{R_i} X_i \leq 1$; so since $R_i \subset Q^3$ implies $\operatorname{dem}_{Q^3} X_i \leq \operatorname{dem}_{R_i} X_i$, we have $\operatorname{dem}_{Q^3} X_i \leq 1$, for $i = 1, 2, \ldots, k$. Now Proposition 2(c) implies $\operatorname{dem}_{Q^3} X \leq 1$.

We combine the next proposition with the preceding one to produce the corollary which is the goal of this section. Although this result is needed mainly for the case $Q^3 = S^3$, the proof does not seem to simplify much in the special case.

PROPOSITION 4. *Suppose* Q^3 *is a compact PL 3-manifold with fixed metric. Then, for each* $\varepsilon > 0$, *there is a* $\delta > 0$ *such that if* $X_1, X_2, \ldots, X_r$ *are disjoint compact subpolyhedra of* Int Q^3 *each of dimension at most one and each of diameter less than* δ, *then there is a connected open* ε*-mapping cylinder neighborhood* G *with a 1-spine in* Int Q^3 *such that* $\bigcup X_i \subset G$ *and each* X_i *contracts to a point in a subset of* G *of diameter less than* ε.

The idea of the proof is to take a triangulation T of Q^3 of small mesh, and to choose δ so small that each X_i lies in the interior of a small PL 3-cell C_i in Q^3. In each C_i we judiciously form a "singular cone" over X_i so that distinct singular cones intersect nicely. We then put T^1 in general position with respect to the collection of singular cones. Finally we pipe $|T^1|$ entirely off the collection of singular cones. In removing an intersection point of $|T^1|$ with one singular cone, it may be necessary to push other singular cones out of the way keeping their bases fixed. G is chosen to be the complement of $|T^1| \cup \partial Q$. Then G is an open ε-mapping cylinder neighborhood of an appropriate subcomplex of the 1-skeleton of a first derived subdivision of T, and G contains all the singular cones.

Proof. Let T be a triangulation of Q^3 of mesh less than $\varepsilon/3$. Let $\delta > 0$ be so that any subset of Int Q^3 of diameter less than δ lies in the interior of a PL 3-cell in Q^3 of diameter less than $\varepsilon/3$.

Suppose $X_1, X_2, \ldots, X_r$ are disjoint compact subpolyhedra of Int Q^3 each of dimension at most 1 and each of diameter less than δ. For each $i = 1, 2, \ldots, r$, there is a PL 3-cell C_i in Q^3 of diameter less than $\varepsilon/3$ such that $X_i \subset \operatorname{Int} C_i$. If $v \in C_i$, let $v * X_i$ denote the set obtained by joining v to the points of X_i by straight line segments in the linear structure of C_i. For each $i = 1, 2, \ldots, r$ we can successively choose a point $v_i \in \operatorname{Int} C_i$ and a triangulation K_i of X_i which is linear in the linear structure of C_i such that:

(i) If α and β are distinct 1-simplices of K_i, then

$$(v_i * \operatorname{Int} \alpha) \cap (v_i * \operatorname{Int} \beta) = \{v_i\};$$

(ii) $(v_i * X_i) \cap X_j$ is a finite set of points for $1 \leq j \leq r, j \neq i$;
(iii) $(v_i * X_i) \cap (v_j * X_j)$ is a subpolyhedron of $v_i * X_i$ of dimension at most 1 for $1 \leq j \leq i - 1$.

General position techniques provide a small PL homeomorphism h_1 of Q^3 such that:

(i) $h_1(|T^1|) \cap \bigcup_{i=1}^r [X_i \cup (v_i * |K_i^0|) \cup \bigcup_{j=1}^{i-1} (v_i * X_i) \cap (v_j * X_j)] = \emptyset$;
(ii) $h_1(|T^0|) \cap \bigcup_{i=1}^r (v_i * X_i) = \emptyset$;
(iii) $h_1(|T^1|) \cap \bigcup_{i=1}^r (v_i * X_i)$ is a finite set of points, at each of which $h_1(|T^1|)$ pierces $\bigcup_{i=1}^r (v_i * X_i)$;
(iv) $h_1(T) = \{h_1(\alpha)\colon \alpha \in T\}$ is a triangulation of Q^3 of mesh less than $\varepsilon/3$;
(v) $h_1 =$ identity on ∂Q^3.

For $i = 1, 2, \ldots, r$, let

$$h_1(|T^1|) \cap (v_i * X_i) = \{p_{i1}, \ldots, p_{is(i)}\}.$$

Then for $1 \leq k \leq s(i)$, there is a 1-simplex $\alpha_{ik} \in K_i$ such that

$$p_{ik} \in \operatorname{Int} (v_i * \alpha_{ik}).$$

Now we can find a *disjoint* collection

$$\{\lambda_{ik}\colon 1 \leq i \leq r,\ 1 \leq k \leq s(i)\}$$

of PL arcs in Q^3 satisfying:

(i) $\operatorname{Int} \lambda_{ik} \subset \operatorname{Int} (v_i * \alpha_{ik})$; p_{ik} is one endpoint of λ_{ik}; and the other endpoint lies in $\operatorname{Int} \alpha_{ik}$,
(ii) $\lambda_{ik} \cap h_1(|T^1|) = \{p_{ik}\}$;
(iii) $\lambda_{ik} \cap X_j = \emptyset$ for $1 \leq j \leq r, j \neq i$;
(iv) $\lambda_{ik} \cap (v_j * X_j)$ is a finite set of points not containing p_{ik} for $1 \leq j \leq r$, $j \neq i$.

Each λ_{ik} serves as the core of a pipe P_{ik}. Indeed, we can construct a *disjoint* collection

$$\{P_{ik}\colon 1 \leq i \leq r;\ 1 \leq k \leq s(i)\}$$

of PL 3-cells in Q^3 such that:

(i) $\lambda_{ik} \subset \operatorname{Int} P_{ik}$;
(ii) $P_{ik} \subset \operatorname{Int} C_i$;
(iii) $P_{ik} \cap (v_i * X_i) \subset (v_i * \alpha_{ik}) - (v_i * \partial\alpha_{ik})$;
(iv) $P_{ik} \cap X_j = \emptyset$ for $1 \leq j \leq r, j \neq i$;

(v) there is a PL homeomorphism of quintuples from

$$(P_{ik}, \lambda_{ik}, P_{ik} \cap (v_i * X_i), P_{ik} \cap \alpha_{ik}, P_{ik} \cap h_1(|T^1|))$$

to

$$([0, 3] \times [-1, 1] \times [-1, 1], [1, 2] \times \{0\} \times \{0\}, [0, 2] \times [-1, 1] \times \{0\}, \{2\} \times [-1, 1] \times \{0\}, \{1\} \times \{0\} \times [-1, 1]).$$

Consequently there is a PL homeomorphism g_{ik} of P_{ik} such that

(vi) $g_{ik}(P_{ik} \cap h_1(|T^1|) \cap (v_i * X_i) = \emptyset$ and

(vii) g_{ik} = identity on ∂P_{ik}.

Define the PL homeomorphism h_2 of Q^3 by

$$h_2 = \begin{cases} g_{ik} & \text{on } P_{ik} \text{ for } 1 \leq i \leq r, 1 \leq k \leq s(i), \\ \text{identity} & \text{on } Q^3 - \bigcup \{\text{Int } P_{ik}: 1 \leq i \leq r, 1 \leq k \leq s(i)\}. \end{cases}$$

For each $i = 1, 2, \ldots, r$, define the PL homeomorphism g_i of Q^3 by

$$g_i = \begin{cases} g_{jk} & \text{on } P_{jk} \text{ for } 1 \leq j \leq r, j \neq i, 1 \leq k \leq s(j). \\ \text{identity} & \text{on } Q^3 - \bigcup \{\text{Int } P_{jk}: 1 \leq j \leq r, j \neq i, 1 \leq k \leq s(j)\}. \end{cases}$$

Then:

(i) h_2 and each g_i $(1 \leq i \leq r)$ are PL $\varepsilon/3$-homeomorphisms of Q^3 which are the identity on ∂Q^3. Hence $h_2 h_1(T) = \{h_2 h_1(\alpha): \alpha \in T\}$ is a triangulation of Q^3 of mesh less than ε.

(ii) For $1 \leq i \leq r$, $g_i(X_i) = X_i$; thus X_i contracts to a point in $g_i(v_i * X_i)$ and diam $g_i(v_i * X_i) < \varepsilon$.

(iii) For $1 \leq i \leq r$, $g_i(v_i * X_i) \cap (\partial Q^3 \cup h_2 h_1(|T^1|)) = \emptyset$.

Observe that we need to shift $v_i * X_i$ to $g_i(v_i * X_i)$. For if $j \neq i$ and $1 \leq k \leq s(j)$, then even though $h_1(|T^1|)$ misses $v_i * X_i$ inside P_{jk}, nevertheless $h_2 \mid P_{jk}$ may push $h_1(|T^1|)$ onto $v_i * X_i$. However $h_2 h_1(|T^1|)$ misses $g_i(v_i * X_i)$ inside P_{jk} because $h_2 \mid P_{jk} = g_i \mid P_{jk}$.

Let $T^* = \{\alpha \in T$: either dim $\alpha \leq 1$ or $\alpha \subset \partial Q^3\}$. Then

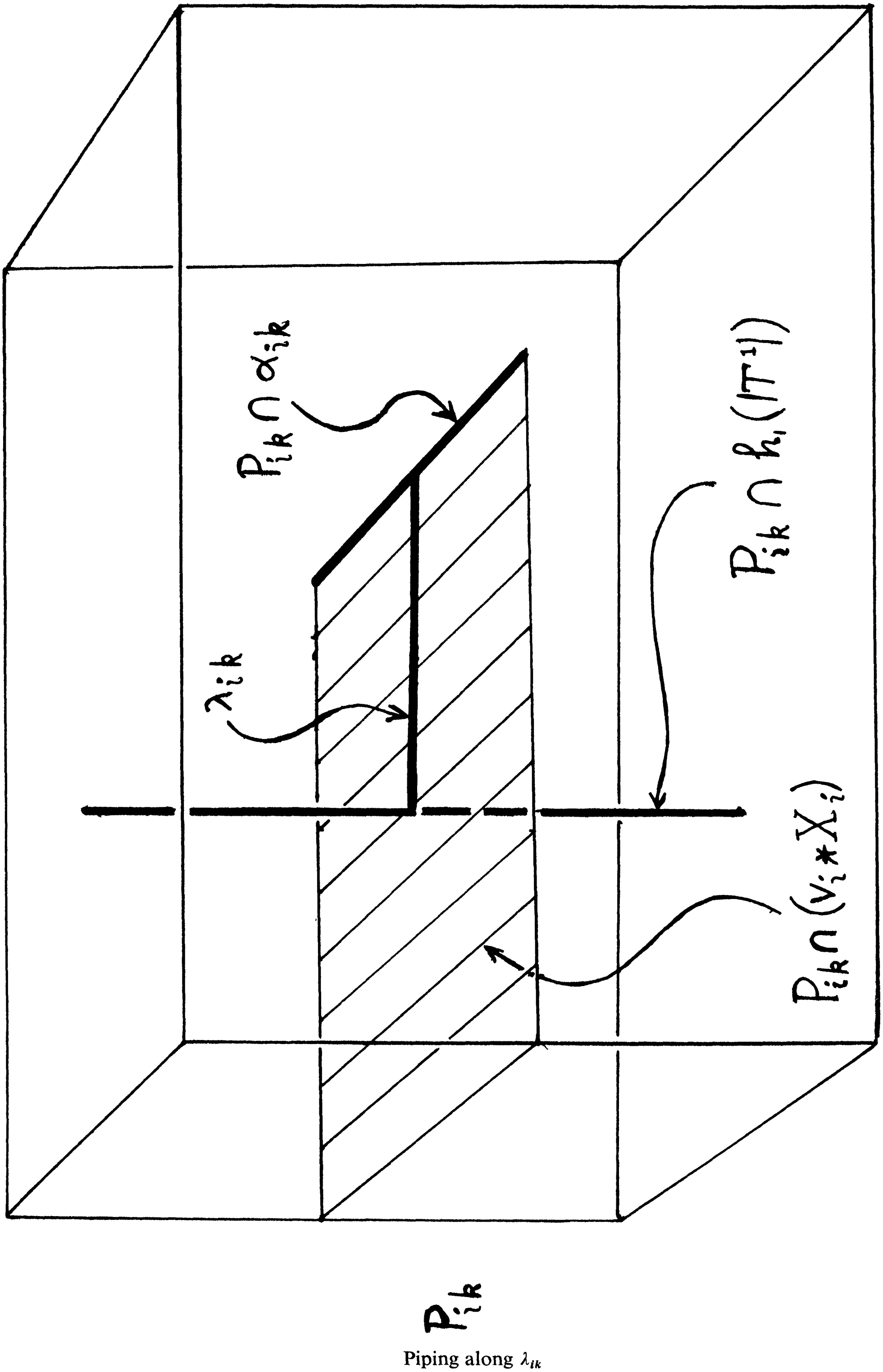
$$g_i(v_i * X_i) \cap h_2 h_1(|T^*|) = \emptyset \quad \text{for } 1 \leq i \leq r.$$

If T' is a first derived subdivision of T and

$$T_* = \{\alpha \in T': \alpha \cap |T^*| = \emptyset\}$$

—the subcomplex of T' which is dual to T^*—then $G = Q^3 - h_2 h_1(|T^*|)$ is an open ε-mapping cylinder neighborhood with a 1-spine $h_2 h_1(|T_*|)$ in Int Q^3. Moreover, for $1 \leq i \leq r$, $X_i \subset g_i(v_i * X_i) \subset G$, X_i contracts to a point in $g_i(v_i * X_i)$ and diam $g_i(v_i * X_i) < \varepsilon$.

Finally we combine the previous two propositions to produce the main result of this section.



P_{ik}

Piping along λ_{ik}

COROLLARY 5. *Suppose Q^3 is a compact PL 3-manifold with fixed metric. Then, for each $\varepsilon > 0$, there is a $\delta > 0$ such that if $X_1, X_2, \ldots, X_r$ are disjoint compact topologically planar subsets of* Int Q^3 *each of dimension at most* 1 *and each of diameter less than δ, then there is an open ε-mapping cylinder neighborhood G with a* 1*-spine in* Int Q^3 *such that $\bigcup X_i \subset G$ and X_i contracts to a point in a subset of G of diameter less than ε.*

Proof. Given $\varepsilon > 0$, Proposition 4 supplies a $0 < \delta < \varepsilon/4$ such that if $Y_1, Y_2, \ldots, Y_s$ are disjoint compact subpolyhedra of Int Q^3 each of dimension at most 1 and each of diameter less than δ, then there is an open $\varepsilon/4$-mapping cylinder neighborhood H with a 1-spine in Int Q^3 such that for $i = 1, 2, \ldots, s$, $Y_i \subset H$ and Y_i contracts to a point in a subset of H of diameter less than $\varepsilon/4$. Suppose $X_1, X_2, \ldots, X_r$ are disjoint compact topologically planar subsets of Int Q^3 each of dimension at most 1 and each of diameter less than δ. Then there are compact PL 3-manifolds $Q_1^3, Q_2^3, \ldots, Q_r^3$ embedded disjointly as PL submanifolds of Q^3 such that for $i = 1, 2, \ldots, r$, $X_i \subset$ Int Q_i^3 and diam $Q_i^3 < \delta$. Proposition 3 implies $\text{dem}_{Q_i^3} X_i \leq 1$ for $i = 1, 2, \ldots, r$. Thus for each $i = 1, 2, \ldots, r$, X_i is contained in an open δ-mapping cylinder neighborhood G_i with a 1-spine Y_i in Int Q_i^3. Then $Y_1, Y_2, \ldots, Y_r$ are disjoint compact subpolyhedra of Int Q^3 each of dimension at most 1 and each of diameter less than δ. It follows that there is an open $\varepsilon/4$-mapping cylinder neighborhood H with a 1-spine Z in Int Q^3 such that for $i = 1, 2, \ldots, r$, $Y_i \subset H$ and Y_i contracts to a point in a subset $\tilde{Y}_i$ of H of diameter less than $\varepsilon/4$.

For each $i = 1, 2, \ldots, r$, if T_i denotes the track of the homotopy pulling X_i down the fibers of G_i into Y_i, then there is a fiber-preserving homeomorphism g_i of G_i which is fixed on Y_i and outside a compact neighborhood of Y_i in G_i such that $T_i \subset g_i(G_i \cap H)$. Define the homeomorphism g of Q^3 by

$$g = \begin{cases} g_i & \text{on } G_i \quad \text{for } i = 1, 2, \ldots, r. \\ \text{identity} & \text{on } Q^3 - \bigcup_{i=1}^{r} G_i. \end{cases}$$

Then g is an $\varepsilon/4$-homeomorphism of Q^3 which fixes

$$Y_1 \cup Y_2 \cup \cdots \cup Y_r \cup \partial Q^3.$$

Thus $g(H)$ is an open $3\varepsilon/4$-mapping cylinder neighborhood of $g(Z)$ in Int Q^3 such that for $i = 1, 2, \ldots, r$, $X_i \subset T_i \cup g(\tilde{Y}_i)$, X_i contracts to a point in $T_i \cup g(\tilde{Y}_i)$ and

$$\text{diam}\,(T_i \cup g(\tilde{Y}_i)) < \varepsilon.$$

Unfortunately, $g(Z)$ may not be a subpolyhedron of Q^3. We remedy this by invoking Theorem 3 of [1] to obtain a PL homeomorphism $g'\colon H \to g(H)$ such that $d(g \mid H, g') < \varepsilon/8$. Then $g'(Z)$ is a subpolyhedron of Q^3. It follows that $g'(H)$ is an open ε-mapping cylinder neighborhood with a 1-spine $g'(Z)$ in Int Q^3, and for each $i = 1, 2, \ldots, r$, $X_i \subset g'(H)$ and X_i contracts to a point in the subset $T_i \cup g(\tilde{Y}_i)$ of $g'(H)$ of diameter less than ε.

3. Proof of theorem 1

We return to the situation and notation of the introduction, with $n = 4$. Our goal is to establish:

THEOREM 1. *Let Σ be a 3-sphere topologically embedded in S^4. Let U be a complementary domain of Σ, with closure C. Then there is a σ-compact $T \subset \Sigma$ such that* $\operatorname{dem}_\Sigma T \leq 1$ *and $U \cup T$ is* 1-ULC.

Let $\mathscr{C}$ denote the separable metric space of all maps

$$f: (\Delta^2, \partial\Delta^2) \to (C, U)$$

with the supremum metric. Let

$$\mathscr{D} = \{f \in \mathscr{C}: \operatorname{dem}_\Sigma (f(\Delta^2) \cap \Sigma) \leq 1\}.$$

The following lemma implies that $\mathscr{D}$ is a dense subset of $\mathscr{C}$. So we may select a countable subset $\{f_1, f_2, f_3, \dots\}$ of $\mathscr{D}$ which is dense in $\mathscr{C}$. Let $T = \bigcup_{i=1}^{\infty} (f_i(\Delta^2) \cap \Sigma)$. Then U is 1-ULC in $U \cup T$. Hence Proposition 1(e) implies $U \cup T$ is 1-ULC, while Proposition 2(c) implies $\operatorname{dem}_\Sigma T \leq 1$. The proof of Theorem 1 is complete modulo a few lemmas.

THE DENSITY LEMMA. *$\mathscr{D}$ is a dense G_δ subset of $\mathscr{C}$.*

If for $i = 1, 2, 3, \dots$, $\mathscr{U}_i$ denotes the set of all $f \in \mathscr{C}$ such that $f(\Delta^2) \cap \Sigma$ lies in an open $1/i$-mapping cylinder neighborhood with a 1-spine in Σ, then $\mathscr{U}_i$ is an open subset of $\mathscr{C}$, and $\mathscr{D} = \bigcap_{i=1}^{\infty} \mathscr{U}_i$.

Let f be a map in $\mathscr{C}$ and let $\varepsilon > 0$. We must construct a map f' in $\mathscr{C}$ such that $d(f, f') < \varepsilon$ and $\operatorname{dem}_\Sigma (f'(\Delta^2) \cap \Sigma) \leq 1$. To obtain f', we construct three sequences: $f_0, f_1, f_2, \dots$, where each f_i is a map in $\mathscr{C}$; $\varepsilon_0, \varepsilon_1, \varepsilon_2, \dots$, where each $\varepsilon_i > 0$; and $G_0, G_1, G_2, \dots$, where each G_i ($i \neq 0$) is an open ε_i-mapping cylinder neighborhood with a 1-spine in Σ. The sequences $\{f_i\}$, $\{\varepsilon_i\}$, and $\{G_i\}$ satisfy

(i) $f_0 = f$;
(ii) $G_0 = \Sigma$;
(iii) $\varepsilon_0 = \varepsilon$;

and for $i = 1, 2, 3, \dots$

(iv) $\varepsilon_i = \frac{1}{2} \min \{\varepsilon_{i-1}, d(f_{i-1}(\Delta^2), \Sigma - G_{i-1})\}$;
(v) $d(f_{i-1}, f_i) < \varepsilon_i$; and
(vi) $f_i(\Delta^2) \cap \Sigma \subset G_i$.

Once we have these sequences, (i) through (vi) imply that there is a map f' in $\mathscr{C}$ defined by $f' = \lim_{i \to \infty} f_i$ such that $d(f, f') < \varepsilon$ and $f'(\Delta^2) \cap \Sigma \subset G_i$ for $i = 1, 2, 3, \dots$. Thus, $\operatorname{dem}_\Sigma (f'(\Delta^2) \cap \Sigma) \leq 1$.

Clearly the following lemma is exactly the tool needed to perform the construction of the sequences $\{G_i\}$, $\{f_i\}$, and $\{\varepsilon_i\}$ inductively.

THE APPROXIMATION LEMMA. *If f is a map in $\mathscr{C}$ and $\varepsilon > 0$, then there is a map f' in $\mathscr{C}$ and an open ε-mapping cylinder neighborhood G with a 1-spine in Σ such that $d(f, f') < \varepsilon$ and $f'(\Delta^2) \cap \Sigma \subset G$.*

Proof. Invoke Corollary 5 to obtain a $\delta > 0$ so that if $X_1, X_2, \ldots, X_r$ are disjoint, compact topologically planar subsets of Σ each of dimension at most 1 and each of diameter less than δ, then there is an open $\varepsilon/3$-mapping cylinder neighborhood G with a 1-spine in Σ such that for $i = 1, 2, \ldots, r$, $X_i \subset G$ and X_i contracts to a point in a subset of G of diameter less than $\varepsilon/3$.

There is a complex K triangulating Δ^2 and there is a general position map

$$g : (\Delta^2, \partial\Delta^2) \to (S^4, U)$$

such that:

(i) $d(f, g) < \varepsilon/3$;
(ii) if $A \in K$, then $\operatorname{diam} g(A) < \min\{\delta, \varepsilon/3\}$;
(iii) $g(|K^1|) \subset U$; and
(iv) $S = \{x \in g(\Delta^2) : g^{-1}(x) \text{ is not a singleton}\}$ is a finite set of points and $\Sigma \cap S = \emptyset$.

K and g are obtained via a sequence of small modifications of f. First choose K to be a triangulation of Δ^2 of mesh so fine that $\operatorname{diam} f(A) < \min\{\delta, \varepsilon/3\}$ for each $A \in K$. Then use Proposition 1(c) to pull $f(|K^1|)$ slightly into U. Take a close general position approximation (into S^4) to the resulting map, and push its singularities (which are a finite number of points) off Σ by a very small homeomorphism of S^4. It is understood that each successive modification of the map must be small enough to preserve the progress made in previous modifications.

Let $X = g^{-1}(S^4 - U)$. Then $X \subset \Delta^2 - |K^1|$, $\dim \operatorname{Bd} X \leq 1$, and g embeds $\operatorname{Bd} X$ in Σ. Hence

$$\{g(A \cap \operatorname{Bd} X) : A \in K \text{ and } A \cap X \neq \emptyset\}$$

is a finite disjoint collection of (nonempty) compact topologically planar subsets of Σ each of dimension at most 1 and each of diameter less than δ. So there is an open $\varepsilon/3$-mapping cylinder neighborhood G with a 1-spine in Σ such that if $A \in K$ and $A \cap X \neq \emptyset$, then $g(A \cap \operatorname{Bd} X) \subset G$ and $g(A \cap \operatorname{Bd} X)$ contracts to a point in a subset of G of diameter less than $\varepsilon/3$. It follows that for each $A \in K$ with $A \cap X \neq \emptyset$, there is an open subset H_A of G of diameter less than $\varepsilon/3$ and there is a map

$$\phi_A : \{(A \cap \operatorname{Bd} X) \times [0, 1]\} \cup \{(X \cap A) \times \{1\}\} \to H_A$$

such that

$$\phi_A(x, 0) = g(x) \quad \text{for each } x \in A \cap \operatorname{Bd} X$$

and $\phi_A((X \cap A) \times \{1\})$ is a singleton. Since H_A is an ANR, Borsuk's homotopy extension theorem provides a map

$$\psi_A : (X \cap A) \times [0, 1] \to H_A$$

such that

$$\psi_A \mid \{(A \cap \operatorname{Bd} X) \times [0, 1]\} \cup \{(X \cap A) \times \{1\}\} = \phi_A.$$

Define $f' \in \mathscr{C}$ by

$$f'(x) = \begin{cases} \psi_A(x, 0) & \text{if } A \in K \text{ and } x \in A \cap X. \\ g(x) & \text{if } x \in \Delta^2 - \operatorname{Int} X. \end{cases}$$

If $A \in K$ and $A \cap X \neq \emptyset$, then

$$\operatorname{diam} g(X \cap A) < \varepsilon/3, \quad \operatorname{diam} \psi_A((X \cap A) \times \{0\}) < \varepsilon/3$$

and

$$g(A \cap \operatorname{Bd} X) = \psi_A((A \cap \operatorname{Bd} X) \times \{0\});$$

therefore $d(g, f') < 2\varepsilon/3$. Consequently $d(f, f') < \varepsilon$. Since

$$f'(\Delta^2 - X) = g(\Delta^2 - X) \subset U,$$

then $f'(\Delta^2) \cap \Sigma \subset f'(X) \subset G$. Theorem 1 is proven.

4. Proof of Theorem 2

Again we return to the scene of the introduction. We assume that T is a subset of Σ such that $U \cup T$ is 1-ULC. We must show:

THEOREM 2. *If $n \geq 6$ and $\operatorname{dem}_\Sigma T \leq 1$, then there is a σ-compactum T' in Σ with $\operatorname{dem}_\Sigma T' \leq 0$ for which $U \cup T'$ is* 1-ULC.

Proof. By Proposition 1(f) we can sssume that T is σ-compact. Theorem 3 of [9] shows that it suffices to exhibit a triangulation Q of Σ of arbitrarily small mesh for which U is 1-ULC in $U \cup (\Sigma - |Q^2|)$. Moreover, this triangulation need not be PL in some given PL structure on Σ. Since U is 1-ULC in $U \cup T$, it suffices to find triangulations of Σ of arbitrarily small mesh whose 2-skeleta miss T. To this end, let $\varepsilon > 0$ and let Q be a triangulation of Σ of mesh less than $\varepsilon/3$. Since $n \geq 6$ and $\operatorname{dem}_\Sigma T \leq 1$, Proposition 2(b) supplies an $\varepsilon/3$-homeomorphism h of Σ such that $T \cap h(|Q^2|) = \emptyset$. Then $h(Q) = \{h(\alpha) : \alpha \in Q\}$ is a triangulation of Σ of mesh less than ε whose 2-skeleton misses T.

We remark that Theorems 1 and 2 can easily be generalized by replacing Σ and S^n by boundaryless connected PL manifolds M^{n-1} and N^n of dimensions $n - 1$ and n, respectively, where M^{n-1} is topologically embedded as a closed subset of N^n which separates N^n, by substituting "1-LC" for "1-ULC", and by making minor alterations in the proofs to accommodate the lack of compactness.

REFERENCES

1. R. H. BING, *Locally tame sets are tame*, Ann. of Math., vol. 59 (1954), pp. 145–158.
2. ———, *A surface is tame if its complement is* 1-ULC, Trans. A.M.S., vol. 101 (1961), pp. 294–305.
3. ———, *Pushing a 2-sphere into its complement*, Michigan Math. J., vol. 11 (1964), pp. 33–45.

4. J. W. CANNON, ULC *properties in neighborhoods of embedded surfaces and curves in* E^3, Canad. J. Math., vol. 25 (1973), pp. 31–73.
5. A. V. CERNAVSKII, *The equivalence of local flatness and local one-connectedness for embeddings of* $(N-1)$*-dimensional manifolds in N-dimensional manifolds for* $N > 4$, Mat. Sbornik, vol. 91 (1973), pp. 279–286 = Math. USSR Sbornik, vol. 20 (1973), pp. 297–304.
6. ROBERT J. DAVERMAN, *Pushing an* $(n-1)$*-sphere in* S^n *almost into its complement*, Duke Math. J., vol. 39 (1972), pp. 719–723.
7. ———, *On the scarcity of tame disks in certain wild cells*, Fund. Math., vol. 79 (1973), pp. 63–77.
8. ———, *Locally nice codimension one manifolds are locally flat*, Bull. A.M.S., vol. 79 (1973), pp. 410–413.
9. ———, *Approximating polyhedra in codimension one spheres embedded in* S^n *by tame polyhedra*, Pacific J. Math., vol. 51 (1974), pp. 417–426.
10. ROBERT D. EDWARDS, "Dimension theory, I," in *Geometric Topology*, Lecture Notes in Mathematics, no. 438, Springer-Verlag, New York, 1975, pp. 195–211.
11. D. R. MCMILLAN, JR., *Taming Cantor sets in* E^n, Bull. A.M.S., vol. 70 (1964), pp. 706–708.
12. ———, *Non-planar embeddings of planar sets in* E^3, Fund. Math., vol. 84 (1974), pp. 245–254.
13. M. A. ŠTAN'KO, *The embedding of compacta in Euclidean space*, Dokl. Akad. Nauk SSSR, vol. 186 (1969), pp. 1269–1272 = Soviet Math. Dokl., vol. 10 (1969), pp. 758–761.

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