Dynamical Systems as Models for Physical Processes

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Simplicity and regularity are associated with predictability. For example, because the orbit of Earth is simple and regular we can always predict when astronomical winter will come. On the other hand, complexity and irregularity are almost synonymous with unpredictability. The atmosphere, for example, being so complex and irregular is rather unpredictable.

Those who try to explain the world we live in have always hoped that within the complexity and irregularity observed in nature, simplicity would be found behind everything, and that ultimately unpredictable events will become predictable. The fact that complexity and irregularity exist in nature is obvious; we only need to look around us to realize that practically everything is random in appearance. Or is it? Clouds, like many other structures in nature come in an infinite number of shapes. Every cloud is different, yet everyone can recognize a cloud. Clouds must then possess unique features that distinguish them from other structures. The question remains: is the irregularity of things like clouds completely random, or is there some order underlying this irregularity?

Over the past two decades physicists, biologists, mathematicians, and scientists from many other disciplines have developed the science of dynamical systems—chaos, fractals, cellular automata—in order to represent and study complexity in nature. In this tutorial we consider all the evidence and understanding that has been gained from the use of these new tools in providing original insights about physical processes.

FRACTALS

Fractal sets, unlike Euclidean objects, possess no characteristic sizes or length scales [1]. They display detailed structure on all length scales, so that when magnified each small portion reproduces a large portion of the set. This property is called *self-similarity* or *scaling* (scale invariance) and is closely connected to the intuitive notion of dimension. Mathematically, scaling is expressed by a power law of the form $C(r) \propto r^A$, where r represents the scale, $C(r)$ is a statistic at a scale r, and A is related linearly to the fractal dimension, D, which takes on noninteger values. Fractals can be exact or random. Exact fractals are produced by recursive algorithms for example, the Koch snowflake shown in Figure 1 or the famed Sierpinski carpet. Exact fractals possess exact self-similarity. Random fractals are products of recursive algorithms plus noise, and do not possess exact self-similarity. In this case, when a small part is magnified it does not reproduce exactly a larger part, but reproduces the statistical properties of a larger part. In this case $\langle C(r) \rangle \propto r^A$ where the brackets indicate averages. In both cases scaling extends to infinitely small scales. The above formulation provides a general way to calculate fractal dimensions. Define the statistic C and determine its value at various scales, r. Plot the logarithm of $C(r)$ versus the logarithm of r. If the resulting curve is linear over a wide range of scales (scaling), then the slope of that linear part is an estimate of the fractal dimension. In the case of the Koch snowflake, $C(r)$ can be the length of the boundary measured with a yardstick of length r. Alternatively, $C(r)$ can be the number of squares of size r needed to cover the boundary (box-counting).

In cases where the scaling is not uniform (i.e., when shapes are statistically invariant under transformations that scale different coordinates by different amounts), then we do not have self-similarity but *self-affinity*. As in the case of self-similarity, self-affinity can be exact or statistical. Statistical self-affinity is

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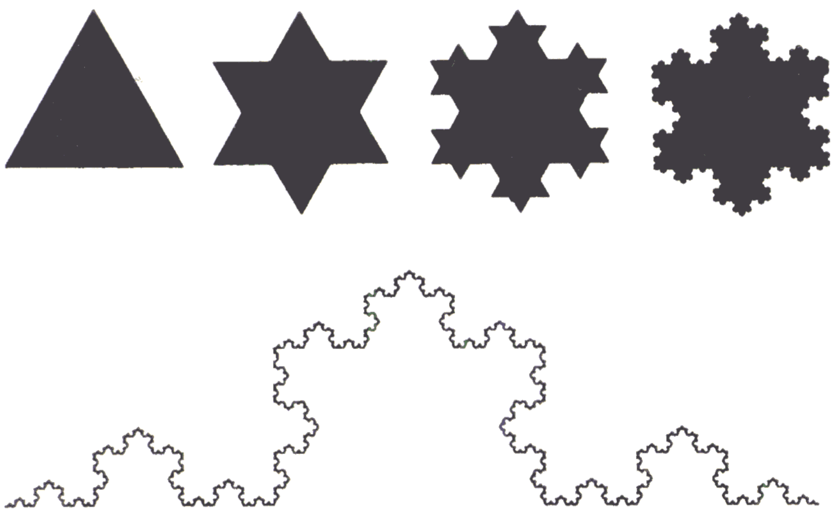
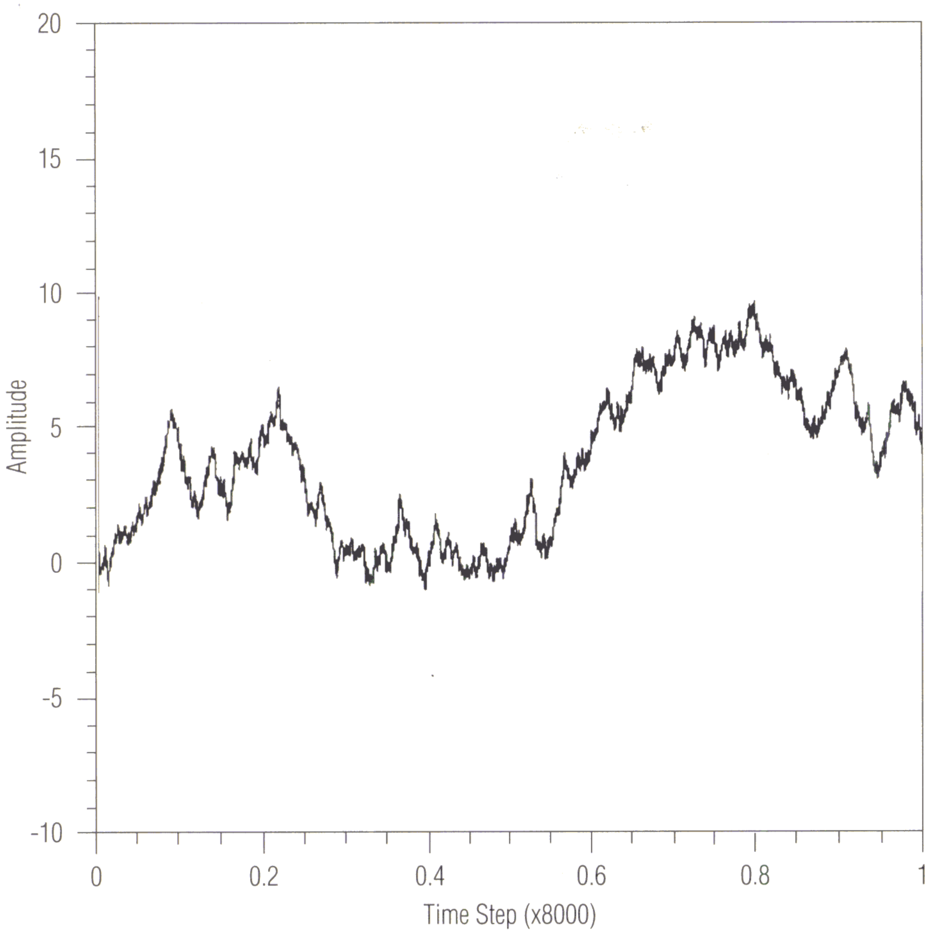
FIGURE 1



Fractals [1] are sets that are not topological. For topological sets the Hausdorff-Besicovitch dimension is an integer (0 for points, 1 for any curve, 2 for surfaces etc.). For fractal sets the Hausdorff-Besicovitch dimension is not an integer but is a real number. Because of that fractals have properties that are beyond topology. The Koch curve or snowflake begins with an equilateral triangle with sides of length one; then at the middle of each side a new equilateral triangle with sides of length one-third is added; and so on. The length of the constructed boundary is 3x4/3x4/3x4/3x........ = ∞. However, that boundary occupies no area at all and it encloses a finite area which is smaller than the area of the circle drawn around the original equilateral triangle. The Hausdorff-Besicovitch dimension of the boundary is 1.2618 (higher than the topological dimension of a curve). Often the Hausdorff-Besicovitch dimension is referred to as the fractal dimension. Such mathematical curiosities, abstract as they seem, have found a place in the study of nonlinear dynamical systems.

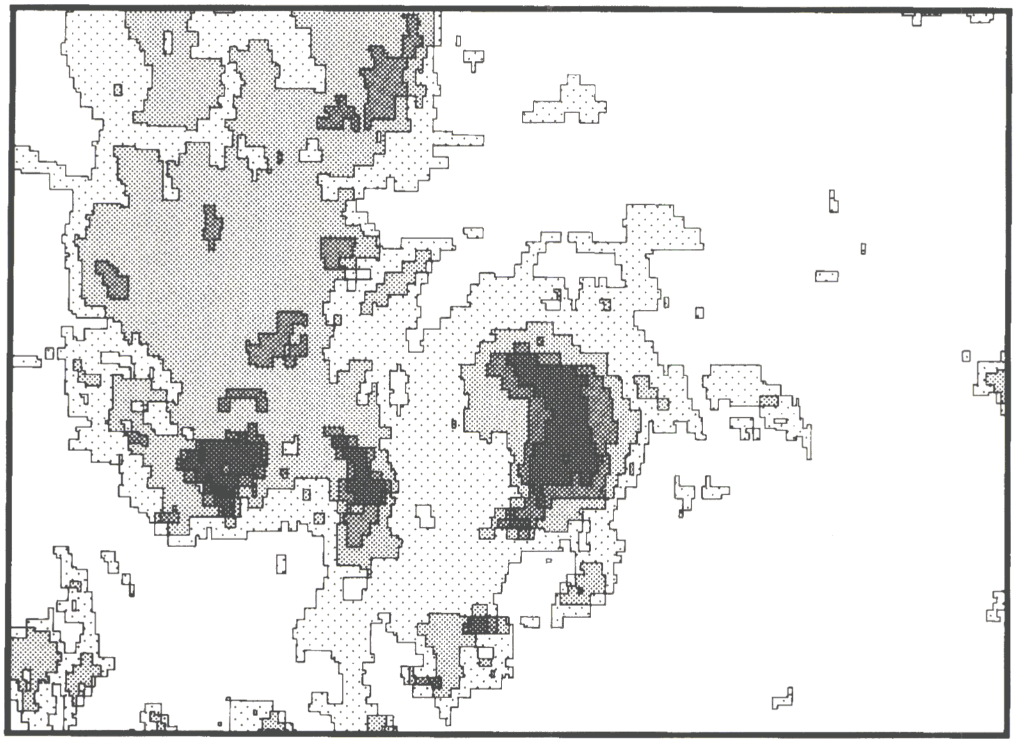
FIGURE 2



The trace of a pure Brownian motion. The amplitude indicates the distance from the origin as a function of the time step.

often the case with noisy time series. Mathematically this is expressed by $\Delta x(\lambda\Delta t) =^d \lambda^H \Delta x(\Delta t)$ for all $\lambda > 0$ where $x(t)$ is the time series and the symbol $=^d$ denotes identity in statistical distributions. This relation dictates that the distribution of increments of x over some time scale $\lambda\Delta t$ is identical to the distribution of increments of x over a lag equal to Δt multiplied by λ^H. Therefore, if time is magnified by a factor λ, x is magnified by a factor λ^H $(0 < H < 1)$. The quantity H characterizes self-affinity in a fashion similar to that by which D measures self-similarity. The value $H = 0.5$ corresponds to the trace of a Brownian motion (see Figure 2), whereas any value $H \neq 0.5$ defines a fractional Brownian motion (fBm) having infinite long-run correlations (either positive if $H > 0.5$ or negative if $H < 0.5$). Note that Brownian motions exhibit spectra of the form $S(f) \propto f^{-a}$ and that their trails have a dimension $D = 1/H$ with $a = 2H + 1$. These properties lie at the heart of computer methods for the generation of random fractal sets of any desired dimension D [2].

FIGURE 3



A rain field delineated at increasing rainfall rate threshold. As the threshold is raised less and less area remains. At each threshold the rain area may exhibit certain fractal properties. In this case the field will be associated with a series of dimensionalities and it is called a multifractal field.

Strictly speaking, a set's defining relation is given by its characteristic function $I(P)$, which can take two values: $I(P) = 1$ (black) if a point P belongs to the set or $I(P) = 0$ (white) if it does not. Natural phenomena can be hardly characterized this way as they demand mathematical objects that allow for the idea of a grey scale. A good example is a rain field, which has an intensity distribution of rainfall rather than being defined as rain—no rain. These objects are called *measures*. The idea of self-similarity is readily extended from sets to measures, in which case these measures are called *multifractals* (they can also be either exact or statistical). The geometric background on which a given physical quantity is distributed can be an ordinary plane, the surface of a sphere, or even a fractal set. Each subset of a measure may exhibit its own fractal dimension. Hence the series of dimensionalities. Figure 3 is such an example. It shows a two-dimensional cross section of a rainfall field, $I(r)$, with a certain intensity distribution (the darker the shading the higher the rainfall rate). For a given intensity, R, the rainfield is defined from the set of points such that $I(r) \geq R$. By increasing R the field is becoming sparser and sparser and (if scaling exists) we obtain a decreasing dimension function $D(R)$ obeying the power law $C(r, R) \propto r^{A(R)}$. Such decreasing function is shown in Figure 4 which for rain shows the derived power law (scaling) at different intensity (dB) thresholds. Note that when dealing with natural systems caution must be exercised in determining the existence and range of scaling. (For more details the reader is advised to consult reference [12].)

CHAOS

Observing the spectra of turbulent motion, one realizes that motion exists at all frequencies with no preferred frequencies. This broad-band structure of the spectrum indicates that the motion is nonperiodic (or, strictly speaking, periodic with an infinite period). Could such a motion be due to a simple nonlinear system? Let us assume that the answer to this question is yes. Then the trajectory in the system's phase space would be nonperiodic and would never cross itself (because once the system returns to a state it was in some time in the past, it must then follow the same path; hence, it is periodic). Thus, the trajectory should be of infinite length, but confined to the finite area defined by its phase space. This can be the case only if the attractor is a fractal set (see Figure 1).

The first such system was discovered in 1963 by Edward Lorenz [3]. This system gives an approximate description of a fluid layer heated from below. The fluid at the bottom gets warmer and rises, creating convection. For a choice of the constants that correspond to sufficient heating, convection may happen in an irregular and turbulent manner. The precise form of the Lorenz system is

$$dx/dt = -ax + ay,$$
$$dy/dt = -xz + bx - y,$$
$$dz/dt = xy - cz,$$

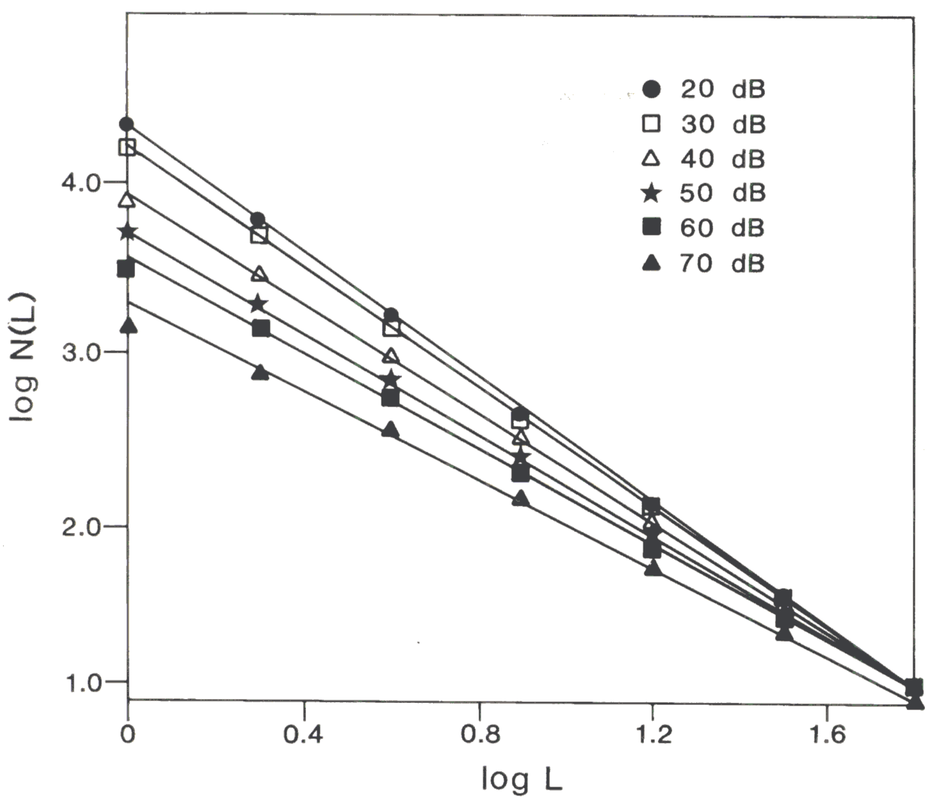
where x is proportional to the intensity of the convective motion, y is proportional to the horizontal temperature variation, z is proportional to the vertical temperature variation, and a, b, c are constants. Figure 5 depicts the path of a trajectory of this system in phase space. Clearly, the Lorenz attractor does not look like a limit cycle (periodic) or a torus (quasi-periodic).

The trajectory is *deterministic* but it is strictly nonperiodic as it irregularly loops to the left and then to the right. Extensive studies have shown that this attractor, as well as other such attractors, possesses a fine structure that is made up of infinitely nested layers (infinite area) occupying zero volume. One may think of it as a Cantor set in higher dimensions. Its fractal (Hausdorff-Besicovitch) dimension has been estimated about to be about 2.06 [4–7].

The fractal nature of an attractor does not only imply nonperiodic orbits. It also causes nearby trajectories to diverge. As with all attractors, trajectories initiated from different initial conditions soon reach the attracting set, but two nearby trajectories do not stay close. They soon diverge and follow totally different paths on the attractor. This divergence is measured by the positive Lyapunov exponents of the system. Attractors with positive Lyapunov exponent(s) are called chaotic attractors. Lyapunov exponents give the rate at which nearby trajectories diverge (positive exponents) or converge (negative exponents). For example, the Lorenz system has one positive Lyapunov exponent equal to 2.16 bits/s. This is interpreted as follows: if an initial point is specified with an accuracy of one part per million (20 bits) its future behavior could not be predicted after about 9 s [20 bits/(2.16 bits/s)] corresponding to about 20 orbits.

Since the system is deterministic, if one knows the initial condition exactly, it is possible to follow the corresponding trajectory and basically predict the evolution of the system forever. Thus, determinism *exists* in chaotic systems. The problem is that we almost never have perfect knowledge of the initial condition. There will always be some deviation of the measured from the actual initial condition. They may be very close to each other but they will not be the same. Thus, even though we may know the laws that govern the evolution of the system exactly, the state of the system at a later time can be totally different from the one predicted by the equations due to the underlying structure of the attractor. Initial errors are amplified and predictability is limited. Furthermore, even if we know the initial condition perfectly, exact computation for long times requires computing values with more and more digits which soon becomes practically impossible. Thus, at some point truncation or round-off error takes place, which introduces a small error that will grow and again lead to unpredictability. Nevertheless, the theory of nonlinear dynamical systems and chaos does provide a new framework to explain the "random looking" character of many observables and to define the limits of predictability of natural systems.

FIGURE 4



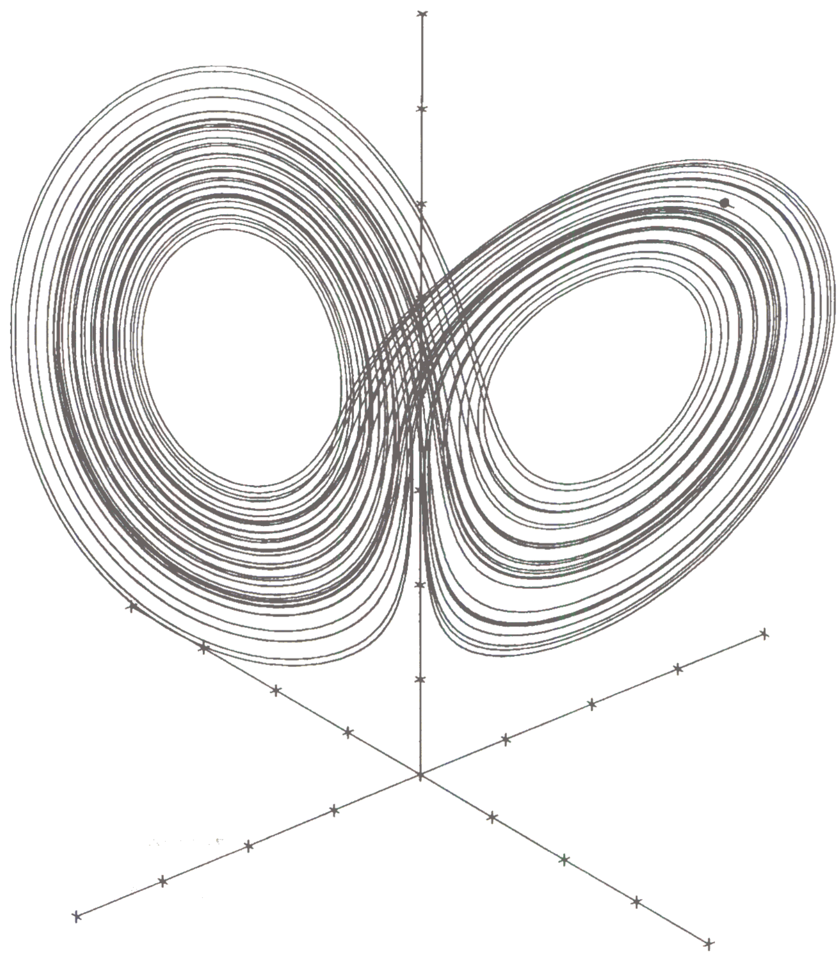
Applying box-counting by considering the rainfield in Figure 3 at different intensities (dB) we obtain the data shown here. In a log-log plot the data suggest scaling which is estimated by a linear regression in the range of scales from $0 \le \log r \le 1.8$. The resulted straight lines have slopes that decreases in magnitude as the threshold increases. This indicates that the rainfield may be a multifractal field.

SUMMARY #1

Our universe is definitely far more complicated than the Lorenz system or the other "standard" chaotic systems reported in the literature. In fact, our universe is an infinitely dimensional system, because it is made up of a practically infinite number of particles not just one as in systems described by a set of ordinary differential equations. In a sense it is naïve to imagine that our universe is described by a grand attractor let alone a low-dimensional attractor. If that were the case, then all observables representing different processes should have the same dimension, which is not suggested from the myriad of reported dimensions. It is possible to make a case that a rather "isolated" subsystem (such as the solar system) may be sufficiently decoupled from its environment so that it obeys its own dynamics. In such cases low-dimensional chaos could be a possibility, with each subsystem having its own attractor. Such subsystems, however, may be rare. That is why most low-dimension estimates reported in the literature do not stand on very firm ground.

In 1989 Tsonis and Elsner [8] suggested that if low-dimensional attractors exist they are associated with subsystems each operating at different space/time scales. In his study on dimension estimates, Lorenz [9] concurs with the suggestion of Tsonis and Elsner [8]. Note that this suggestion does not imply only weakly-coupled subsystems. Strongly-coupled subsystems may very well exist. If so, communication contaminates and may well destroy low-dimensional chaos. Unfortunately, it is not known a priori that a particular observable represents a weakly-coupled subsystem, and, thus, often the procedure and interpretation of the resulting calculation of the attractor dimension is not valid. Therefore, we should not be surprised if a low-dimensional system exists somewhere in nature. But we should not expect low-dimensional chaos everywhere. After all, chaos does not explain the spontaneity and self-organization observed in many natural systems. Self-organization refers to phenomena on a macroscopic scale—a scale often much larger than that of the fundamental interactions—in the form of spatial patterns or temporal rhythms. Examples of self-organization are the Benard convection, hurricanes, developing embryos, and evolutionary ecosystems.

FIGURE 5



A trajectory of the Lorenz system with a = 10, b = 28, and c = 8/3.

Along the same lines, a quick look around us immediately reveals that exact fractals (Figure 1) like exact cubes or exact spheres do not exist in nature. Random fractals, on the other hand, may very well exist. Computer generated random fractals (from algorithms or physically-based models) resemble natural objects quite well [1, 2, 10, 11]. However, for reasons similar to those discussed previously, fractals and scaling may not be as abundant as claimed, and their existence may be restricted to a certain range of scales depending on the "connectivity" of the generating physical system to other systems [12]. As such, fractals, chaos, and scaling become significant if we know exactly over what range of scales they apply, as they can then reveal important connections to dynamics or the rules dictating the different scales.

We thus arrive at the inevitable conclusion that low-dimensional chaos may be limited and that randomness is an integral part of processes in nature. An obvious question then arises: can randomness be linked to self-organization? An answer to this question is provided by the theory of cellular automata.

CELLULAR AUTOMATA

The complexity observed in fractal sets and in chaotic time series from dynamical systems with a few equations shows that simple systems can generate very complex behavior.

In a series of papers [13-16], Wolfram proposed that the origin of such complexity could be investigated through simple mathematical models which are termed *cellular automata*. Models based on cellular automata provide an alternative to differential equations (ordinary or partial) and iterative mappings. They involve discrete coordinates and variables,as well discrete time steps. They can exhibit complex behavior analogous to that found in differential equations. But because they are extremely simple they offer the possibility of a detailed and complete investigation of complexity.

In its simplest form, a one-dimensional cellular automata consists of a line of sites or cells with each site carrying the value of 0 or 1. The value a_i of site i at time t is determined by a simple rule that depends on the previous values of sites in a neighborhood of site i. Explicitly, the rule f giving the value at site i at time t can be written as

$$a_i(t) = f[a_{i-r}(t-1), a_{i-r+1}(t-1), \ldots, a_{i+r}(t-1)] \,.$$

Even for a neighborhood size of radius $r = 1$, very complex behavior can be obtained.

For example, consider the cellular automata for which the value at each site is updated according to the rule

$$a_i(t) = 1, \ \text{if} \sum_{\substack{j=i-1 \\ j\neq i}}^{i+1} a_j(t-1) = 1,$$

$$a_i(t) = 0, \ \text{otherwise.} \qquad (1)$$

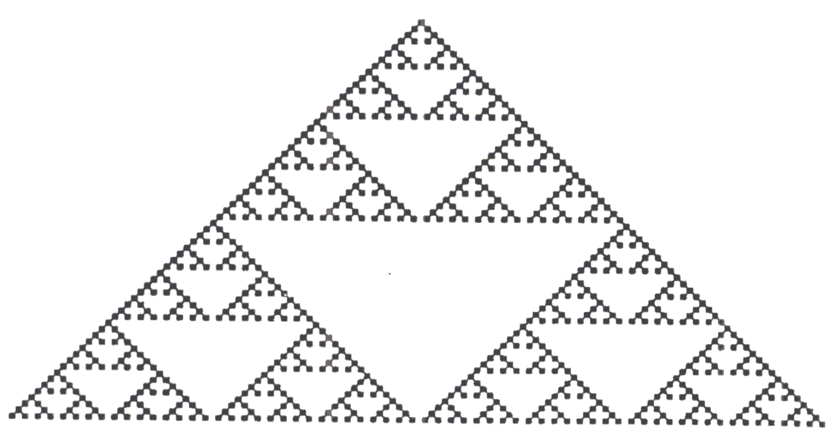
This corresponds to a neighborhood of size $r = 1$. Starting with the *simple* initial state where only one fixed site is allowed to grow

...........0000001000000............

the emerging structure is given in Figure 6. This structure is an exact self-similar (fractal) structure that grows indefinitely, having a dimension of about 1.59 [14]. It is found that patterns generated from such simple fixed initial configurations either die out after some time steps or they evolve to a fixed size or they grow indefinitely (either as exact or random fractals) or they grow and contract chaotically.

When randomness is introduced in cellular automata by means of making the initial state *disordered* (where each site is assigned each of its possible values with an independent equal probability), the patterns generated can be again classified into four distinct categories. The difference, however, is that now the category of exact fractals (that we find in the cases with simple initial states) disappears, while a new category emerges. The four categories now are: 1) spatially homogeneous patterns, 2) a sequence of simple stable or periodic structures, 3) chaotic aperiodic behavior, and 4) complicated localized structures showing self-organization and often propagation (Figure 7). The analogue of these four classes to dynamical systems is obvious. Class 1 corresponds to limit points, class 2 to limit cycles, and class 3 to chaotic attractors. No direct correspondence for class 4 is immediately obvious. But we now know that this class corresponds to the so-called "onset of chaos" or "edge of chaos," which is the region in the parameter space of a system where it begins a transition from a periodic regime to chaotic behavior [17, 18]. This discovery is fundamental to what is now called the "science of complexity." Therefore, when rules *and* randomness are present, exact fractals disappear, while steady states, periodic evolutions and chaos remain, and self-organization appears. Thus, we conclude from this discussion that all phenomena observed in nature can be explained only if randomness is built-into the picture!

FIGURE 6



The evolution of a one-dimensional cellular automaton obeying the rule given by equation (1). Sites with values one are black. Sites with value zero are left white. As the time step increases the structure keeps on growing (from Wolfram 1994).

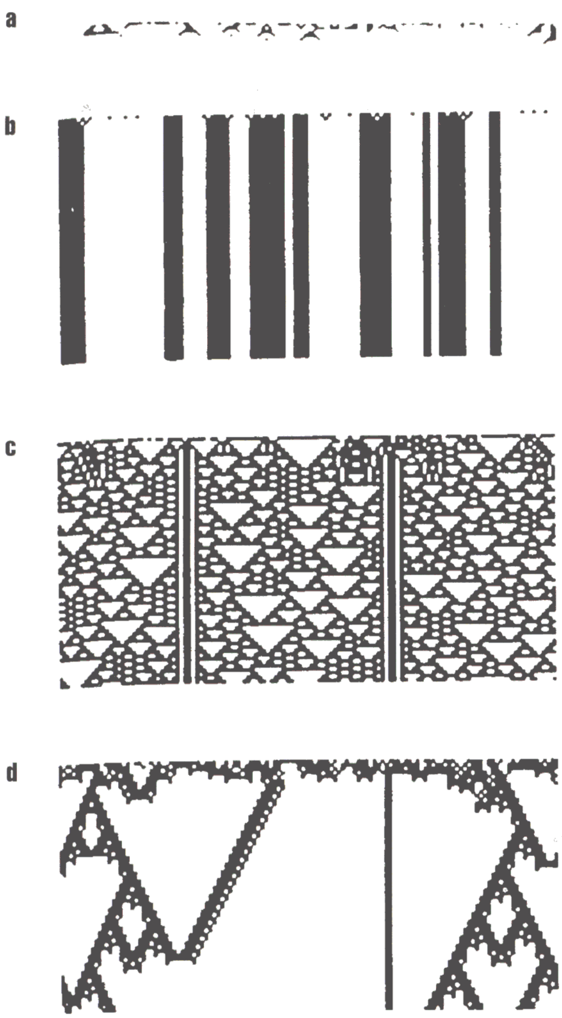
THE ROLE OF RANDOMNESS

Others there are who believe that chance is a cause but that it is inscrutable to human intelligence as being a divine thing and full of mystery.

Aristotle, Physics, Book II,4.

The obvious question that arises next is what is the role of randomness? Why

FIGURE 7



Evolution of various cellular automata from disordered initial states. The four distinct classes discussed in the text are shown [16].

FIGURE 8

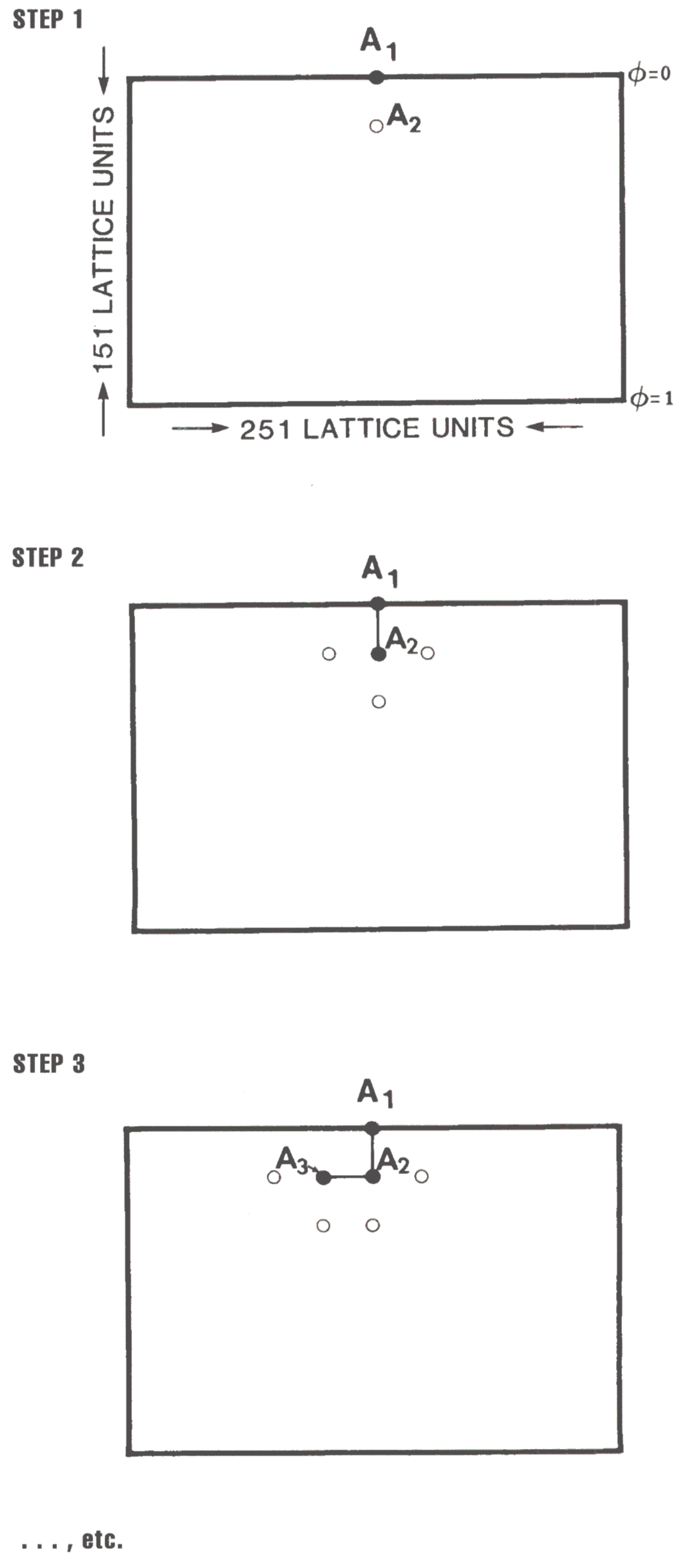
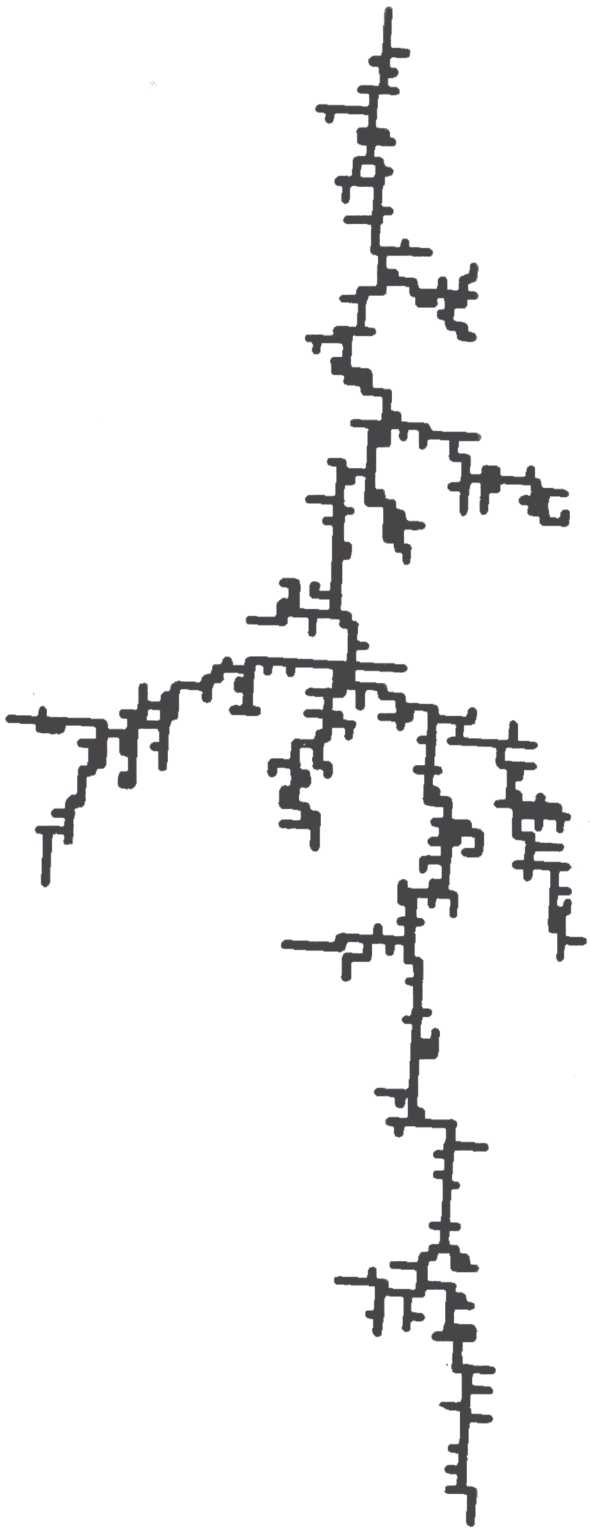


Illustration of the model used to simulate lightning. The discharge pattern is indicated by the black dots connected with solid lines and is considered equipotential. the open circles indicated the possible growth sites. The probability of each one of these sites is proportional to the local potential field (see text for details).

do we have to have randomness? As has been eloquently presented by Prigogine's group [19, 20], randomness is closely connected to irreversibility, which is a fundamental property in everything that happens in our universe. Additional insight into this question comes from experiments. The structure in Figure 9 was generated by a model that is a modification of the nonequilibrium model proposed in [21] for the modeling of two-dimensional radial discharge (Lichtenberg figures). The details of the model [10] used to simulate lightning in the atmosphere are that the simulation is a step-wise procedure carried on a two-dimensional lattice (Figure 8), in which the potential (ϕ) of the top and the bottom row is fixed at values $\phi = 0$ and $\phi = 1$, respectively. Periodic boundary conditions are assumed at the sides of the lattice, with only the middle point of the top row (A_1) being capable of growth. Given these boundary conditions, the potential at every point of the

FIGURE 9



An example of a lightning generated by the model discussed here. This lightning is a random fractal with a dimension of about 1.37.

lattice is obtained by solving the Laplace equation $\nabla^2 \phi = 0$. On a two-dimensional lattice this is obtained by iterating the following equation using successive over relaxation (SOR):

$$\phi_{i,j} = 1/4[\phi_{i+1,j} + \phi_{i-1,j} + \phi_{i,j+1} + \phi_{i,j-1}].$$

All the immediate nonzero neighbors of the point A_1 are then considered as possible candidates, one of which will be added to the evolving structure. The candidates are indicated in Figure 8 by the open circles, while the evolving pattern is given by the black dots. In step 1 there is only one possible candidate. Therefore, point A_2 will be added to the discharge pattern which is considered equipotential ($\phi = 0$). In step 2, one again solves the Laplace equation taking into account that the boundary conditions include the discharge pattern. There are three possible candidates in step 2, each one is assumed to be associated with a "growth" probability p which is defined as

$$p_i = \phi_i^2 / \Sigma \phi_i^2,$$

where $i = 1,....,N$, with N being the number of all possible candidates. Accordingly, at each step a probability distribution is defined. Given this distribution, a candidate is chosen at random and is added to the evolving pattern. The above procedure is then repeated until the discharge reaches the bottom row. An example of such a simulated lightning is shown in Figure 9. This type of modeling is a classic example of how rules (boundary conditions, plus the Laplace equation) and randomness can produce objects very similar to those seen in nature. In fact, the lightning in Figure 9 has a fractal dimension of about 1.37 which is close to the average value of 1.34 obtained from calculations done with many photographs of real lightning in the atmosphere [10].

Another example is snowflakes. The hexagonal symmetry observed in natural snowflakes is due to the molecular structure of ice. The actual growth and shape of a snowflake is subjected to Darcy's law [11] and to environmental effects (noise). Snowflakes have been simulated by a statistical mechanical model [11] that incorporates a parameter producing the six-fold anisotropy of natural flakes, Darcy's law, and random motion of water molecules. Such simulations, unlike the Koch snowflake (Figure 1), which is an exact fractal, produce flakes that are random fractals having a dimension of about 1.5, and display a striking resemblance to natural snowflakes [22].

Since the model gives the probability of every site being occupied at each step, one may calculate the probability of the whole structure by multiplying the probabilities associated with all the selected points. In general, if we assume that at the nth step a structure is made up of n points, denoted $A_1, A_2,....., A_n$, which are selected in that order, then the probability, $P(n)$ of the structure is given by

$$P(n) = P(A_1, A_2, \ldots, A_n) =$$
$$= P(A_1)P(A_2|A_1)P(A_3|A_2A_1)\ldots\ldots P(A_n|A_{n-1}A_{n-2}\ldots\ldots A_1).$$

Experimentation with random fractal lightning, and with "cooked up" exact fractals and non-fractal structures [23] having the same number of points, indicates that a random fractal outcome is 10^{500} (!) times higher than that of a nonfractal or an exact fractal outcome after only 200 steps. This staggering number clearly demonstrates that random fractals are very likely events. When a system is in a steady, nonequilibrium state in the linear thermodynamic regime (i.e., when the system cannot achieve an equilibrium state at which the entropy production becomes zero), the most probable state corresponds to the one of minimum entropy production or to a state of *least dissipation* [20]. Thus, the fact that random fractals are the most probable events provides a direct link between the existence of random fractals and the adaptation of nature toward least dissipation. If chance is eliminated in the lightning model by imposing a rule dictating that the point with the highest growth probability value is always chosen, then we will always end up with a nice Euclidean straight line. Such a result, however, is hardly appropriate for nature since it contributes very little to the discharge process (branching structures remove charge from a large area more effectively than a straight line). Thus, in nonequilibrium systems (in the linear regime), effectiveness and least dissipation require rules and randomness to work together. Far from equilibrium, however, events may be different. In far from equilibrium states a fluctuation within the system may become too big. Then, especially near *bifurcation* points, the fluctuation may drive the average and basically dictate the fate of the system. Such situations are characterized by self-organization. The Benard convection is such a situation. For typical convection we can assume that there are always small fluctuations from the average current, which decay below some critical value of the temperature gradient. Above the critical value, some fluctuations may amplify thus starting a macroscopic current. In effect, we have a large fluctuation taking over the average current and producing a coherent structure (self-organization). Note that the above facts are in complete agreement with the self-organization ob-

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served in the so-called "edge of chaos," or "onset of chaos," which as mentioned above in the presence of randomness (disordered initial states) is the region in the parameter space where the system begins to bifurcate toward chaos.

SUMMARY # 2

It is now easy to see how a general framework for natural processes based on the notion of connected subsystems discussed earlier and on randomness could explain how physical processes emerge. The actual source of randomness is not important. Only the fact that it exists is of significance. The four fundamental forces cause rules and dynamical systems to "crystallize" at various space/time scales. Our solar system, El Niño, a cloud, and an ecosystem are such systems. These systems are usually nonlinear and may exhibit a variety of stable periodic (equilibrium attractors) or chaotic (nonequilibrium attractors) behaviors. All systems are connected with each other, as in a web with various degrees of connectivity. Accordingly, any system can transmit "information" to another system thus perturbing its behavior. This "information" plays the role of an ever present external noise which perturbs the behavior of the system. Depending on the connectivity of the system to the other systems, the effect can be dramatic or negligible. Systems with weak connectivities will be approximately "independent," and as such they may exhibit low-dimensional chaos depending on the parameters involved. Nonlinearity and imperfect initial conditions will make these systems unpredictable after some time. Identification of these subsystems thus becomes important, since it allows us to treat these systems as isolated or closed systems. Otherwise, low-dimensional chaos will not be favored. Instead, spontaneity and self-organization may ensue as external (and possibly internal) causes become important. Similar remarks apply to the existence of scaling.

In this article, we have presented evidence linking randomness and its interplay with dynamical systems (rules) to physical processes. We have shown why this is desirable and how it leads to a conceptual framework for physical processes. Apart from the theoretical and possibly philosophical aspects of our paper, an extremely important issue arises: The prediction of time series. In the past, prediction of random-looking "complex" records exhibiting spectra appropriate to noise were treated purely statistically. Then came chaos, which taught us that random-looking behavior and broad-band spectra may arise from simple deterministic systems. A switch to dynamical-type prediction then followed [24-26]. It is clear from our work, however, that the two cannot always be separated. If rules and randomness coexist then approaches used to model and predict the outcome of physical systems should combine both! We hope that this work will motivate further developments in this area.

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SUGGESTED READING

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GLOSSARY

Attractor:
A dynamical system may be such that its evolution causes it to approach a stable final state. In the phase space representing the system the final state tends to a set of points called the attractor. The attractor may be a point, a line, a surface or a fractal set.

Bifurcation:
A phenomenon whereby the number of solutions of a dynamical system changes suddenly, as one of the parameters of the system crosses a critical value.

Cellular Automata:
Mathematical realizations of physical systems in which space and time are discrete, and physical quantities take on a finite set of discrete values. They are constructed from many identical components, each simple, but together capable of complex behavior.

Chaos:
Random-looking behavior occurring in deterministic nonlinear dynamical systems.

Chaotic dynamics:
Time-depended aperiodic regime arising in nonlinear dynamical systems in which very close initial conditions tend to diverge exponentially.

Complexity:
The science for studying complex systems. A complex system is a system that can display a great diversity of behavior and undergo transitions between different states including self-organization.

Entropy:
A thermodynamic property of a system which corresponds intuitively to the degree of disorder.

Fractal geometry:
Extension of Euclidean geometry for describing irregular and fragmented patterns. Such patterns may be characterized by an noninteger "fractal dimension."

Lyapunov exponent:
A measure of the rate of exponential divergence of initially close states of a dynamical system. In a chaotic system there is at least one positive exponent.

Phase space:
A space whose coordinates are the independent variables describing a dynamical system.

Predictability:
The ability to predict the future state of a system from a predescribed initial state.

Self-organization:
Spontaneous emergence of order, arising when certain parameters of a physical system reach critical values.

Self-similarity:
see scale invariance

Scale-invariance (scaling):
A physical system or object is said to exhibit scaling if its appearance (in a statistical sense) remains unchanged under a magnification or zooming operation.

Strange attractor:
An attractor in the phase space of a dynamical system having fractional dimensionality. Strange attractors are associated with chaotic dynamics.