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The Spatial–Temporal Scaling Properties of Rain and the Validity of Taylor's Hypothesis in the Atmosphere

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The scaling properties of rain field maps in the (x,y,t) space have been investigated, and the elliptical dimension (d_{el}), which characterizes the degree of anisotropy of the field, has been calculated. A value of d_{el} = 3.0 would correspond to statistical isotropy in space and time similar to the "frozen turbulence" hypothesis out forward by G.I Taylor which implies that the space/time transformations are not scale dependent. Departures below this value indicate that the statistical properties in time differ from those in space. Here, using weather radar data, we estimate that d_{el}=2.62 which indicates that the validity of Taylor's hypothesis in the atmosphere may be thought via space/time transformations which are scale dependent.

Свойства просранствено-временного масштаба осадков и действительность гипотези Тейлора в атмосфере Исследованы масштабные свойства карт поля осадков в (x,y,t) пространстве и расчитана эллиптическая размерность, d_{el}, которая характеризирует степень анизотропности поля. Значение d_{el}=3.0 соответствует статистичес- кой изотропности в пространстве и времени, схожую на гипотезу "замороженную турбулентности" Тейлора, которая утверждает, что пространсвенно -временные трансформации не зависят от масштаба. Отклонения ниже этого значения показывают что статистические свойства по времени отличаются от этих в пространстве. Используя данные метеорологического локатора, мы получили значение d_{el}=2.62 что дает возможность представить себе применимость гипотези Тейлора при помощи пространствено- временных трансформаций, которые зависят от масштаба.

Introduction

An object is said to be fractal when its large scale structure is a magnified copy of its small scale structure. In other words, two pieces of a fractal object one of size A and the other size A' (where A' is contained in A) are statisticaly equivalent over a wide range of intermediate lengths as long as the smaller piece is magnified by a factor A/A'. This property is a kind of symmetry and is called scale invariance or simply scaling. The exact relation between the small and large scales is dictated by the so-called fractal dimension. The above described scaling is often referred to as simple scaling and it has been shown to exist in many mathematical and physical systems (Mandelbrot, 1983; Lovejoy, 1982; Tsonis and Elsner 1987).

Lately it has been realized that in nature simple (or single) scaling is actually a special case. In nature we usually deal with fields (in which a number is assigned to each point in space, for example, temperature). Such fields have been found to involve multiple scaling and thus are characterized by a sequence of fractal dimensions (Lovejoy et al., 1987). The notion of multiple fractal dimensions can easily be understood from the following example. Consider the precipitation area. This field is defined at different intensities (i. e., rainfall rates) starting at some minimum detectable signal, R_{min}, and going up to some maximum signal, R_{max}. From this field one may define many subfields by considering the precipitation areas (strictly speaking the fractions) delineated by any intensity $R_{min}<R<R_{max}$. Obviously as R approaches R_{max}, less, more

intense and increasingly sparser rain area will remain. At some point there will be no rain area at all. Then the dimension of the rain field will be zero. Thus, the dimension of the rainfield is a function of its intensity and thus the simple scaling discussed above may only be a special case. The above ideas on multifractals (or multiscaling) can be traced to an earlier work by Mandelbrot (1974).

At a fixed threshold T the fractal dimension of a field embedded in d-Euclidean space can be found by the so-called box-counting algorithm. One covers the field by d-dimensional boxes of side length L and counts the number $N(L)$ of boxes that are needed to completely cover the set (Hentschel and Procaccia, 1983). If one finds that for a wide range of scales $N(L) \alpha L D_d(T)$ then $D_d(T)$ is an estimation of the fractal dimension of the field for that threshold T. These ideas have been explored by Lovejoy et al. (1987) who analyzed the two (x,y) and three (x,y,z) dimensional structure of the rain field (employing radar data). Lately, in a different approach Gupta and Waymire (1990) investigated multiscaling in terms of the moments of instantaneous spatial rainfall.

In addition to the above concepts, there is the notion of the elliptical dimension (d_{el}). Unlike "mathematical" fractals where the iteration process can be extended to all scales, "physical" fractals like those observed in nature may be observed within a range of scales only. Thus, in the atmosphere a fractal process may be terminated by viscosity at small scales and by the finite scale of the planet at large scales. As a matter of fact nature may "interfere" with a fractal process in many interesting ways depending largely on the imposed boundary conditions. Let us think of cloudiness for a minute. Clouds occur in the troposphere which is on the average about 10km high. Thus, cloudiness is restricted between two spheres one of radius 6380km (equatorial radius) and the other of radius 6390km. Obviously the vertical scale is much smaller than the horizontal scales. This stratification obviously does not allow the above discussed simple or isotropic scaling to exist. A small cumulus cloud (smal scale cloudiness) when magnified will not produce the flat structure of cloudiness observed over large scales. One may think of the large scale cloudiness field as a magnified and compressed copy of the small scale cloudiness. In such cases the field could be treated with an anisotropic scaling that might involve magnification and compression and could be characterized by a dimension which is called the elliptical dimension. When we seek to characterize the anisotropy of a field whose degree of stratification is not known a priori the elliptical dimension can be estimated by a procedure called the "elliptical dimensional sampling" and involves repeating the box-counting algorithm with families of stratified boxes of sizes L by L by LH_z for $0.0<H_z<1.0$. This method objectively determines which family of sampling boxes (i.e. optimum H_z) has the same degree of stratification as the field in question. If the optimum H_z is known the elliptical dimension of the field, d_{el}, is determined from the relation $d_{el} = 1 + 1 + H_z$. If the field is iostropic then $H_z = 1.0$ and $d_{el}=3.0$. The field in this case is said to be not stratified at all (the statistical properties are the same along all directions). If the field is completely stratified then $H_z=0.0$ and $d_{el}=2.0$. For the (x,y,z) rain intensity field, Lovejoy et al. (1987) obtained an estimate of $d_{el}=2.22$.

Data Analysis and Results

In this report we analyse the scaling properties of rain in the space/time domain. Specifically we analyze the (x,y,t) rain field which is made up of sequences of radar reflectivities as delineated by Plan Position Indicator (PPI) maps (Hill, 1988). These reflectivities are one of the highest quality geophysical data available for this purpose. The precipitation particles are very effective scatterers that allow the precipitation structure to be sampled quickly and without perturbation The reflectivity, Z is the integrated backscatter of the precipitation particles. This reflectivily is usually converted to rainfall rate R via semi-emperical relations of the form $Z = AR^b$ (were A and b are constants). Here we investigate the structure of the reflectivity field. By studying reflectivities directly, rather than using R, we avoid the traditional radar calibration problem. The data used were recorded at the McGill weather radar observatory on 17 October 1989 over a period of twenty minutes. The temporal resolution of the data (time separation between consecutive PPI maps) is 10 seconds. This is the highest resolution in time we can have. The data were sampled in polar coordinates (r,θ) with 120 by 360 elements. Thus, the complete (r,θ,t) field contains $120\times360\times120 =$ 5 184 000 elements. The intensities are resolved into 256 logarithmic levels that are 1/4 decibels (dB) apart. The entire scale transform spans a range of 240x1/4 = 60dB = a factor of 10^6 in reflectivity intensities. Note that reflectivity levels in rain can exceed the minimum detectable signal (approximately 16dB) by a factor of 10^5.

The field in question is defined at different intesity thresholds. In order to estimate the elliptical dimension of such a field we seek to determine the zero of the function (Lovejoy et al, 1987; Schertzer and Lovejoy, 1987)

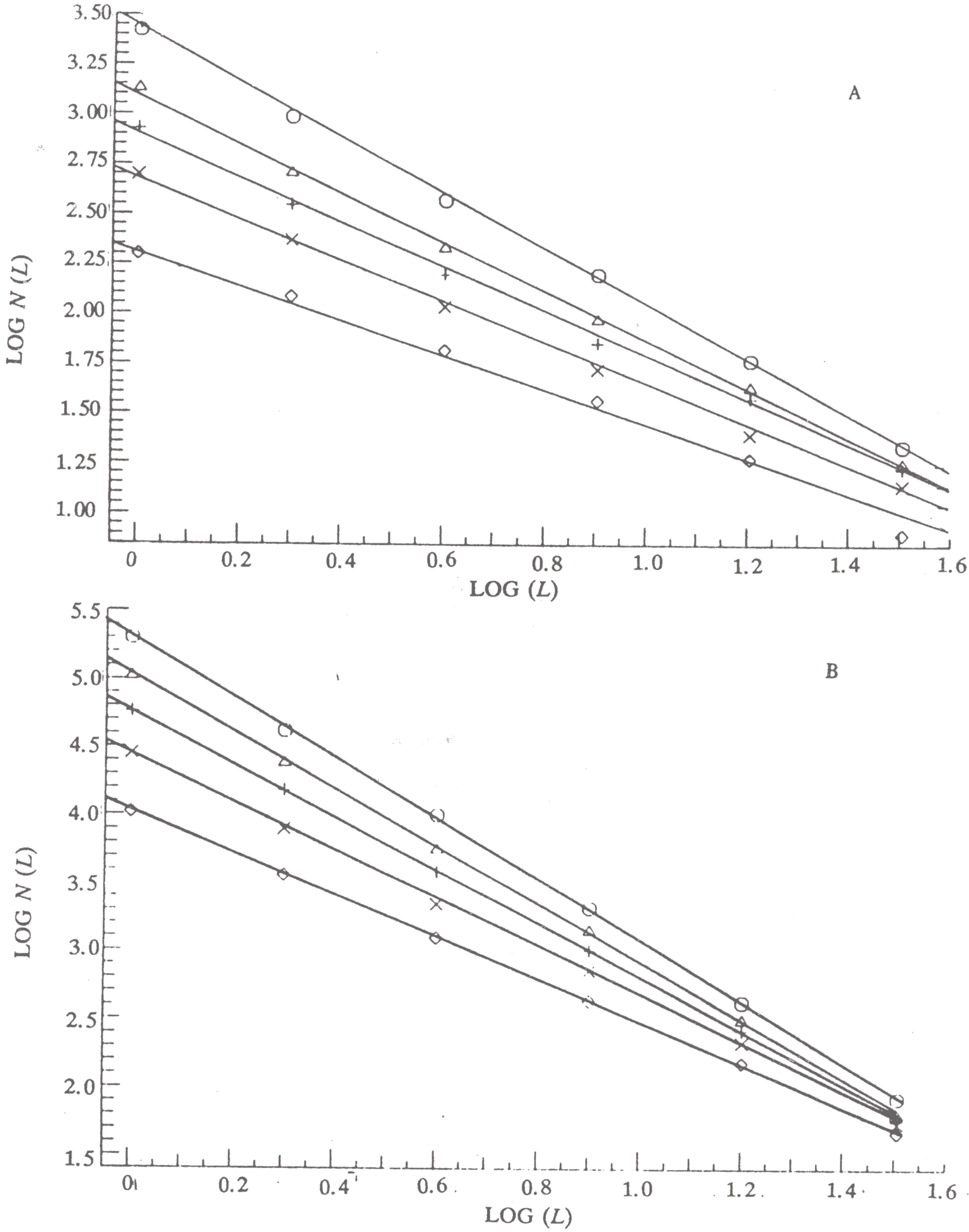

Fig.1. In box-counting we first define our field (as delineated a threshold, T). Then we find the number $N(L)$ of disjoined squares (or boxes of appropriate shape and dimension) of size L needed to completely cover the field. The dimension, $D(T)$, of the field is then estimated by the Eq. $N(L)L - D(T)$. Thus in a $logN(L)$ versus $logL$ plot $D(T)$ can be estimated by the slope. (A) Plot of $logN(L)$ versus $logL$ for five selected radar reflectivity thresholds for a single PPI map. Because of the sampling in polar coordinates the boxes used are in the horizontal direction sectorial (pie-shaped) boxes. The thresholds increased from top to bottom by $10dB$ from **4dB** above the minimum detectable signal to a maximum of 60 dB. The negative slope, $D_2(T)$, decreased from 1.36 to 0.89. (B) A similar analysis but for the complete (r, θ,, t) rainfield. The negative slope $D_3(T)$ decreased from 2.20 to 1.55.

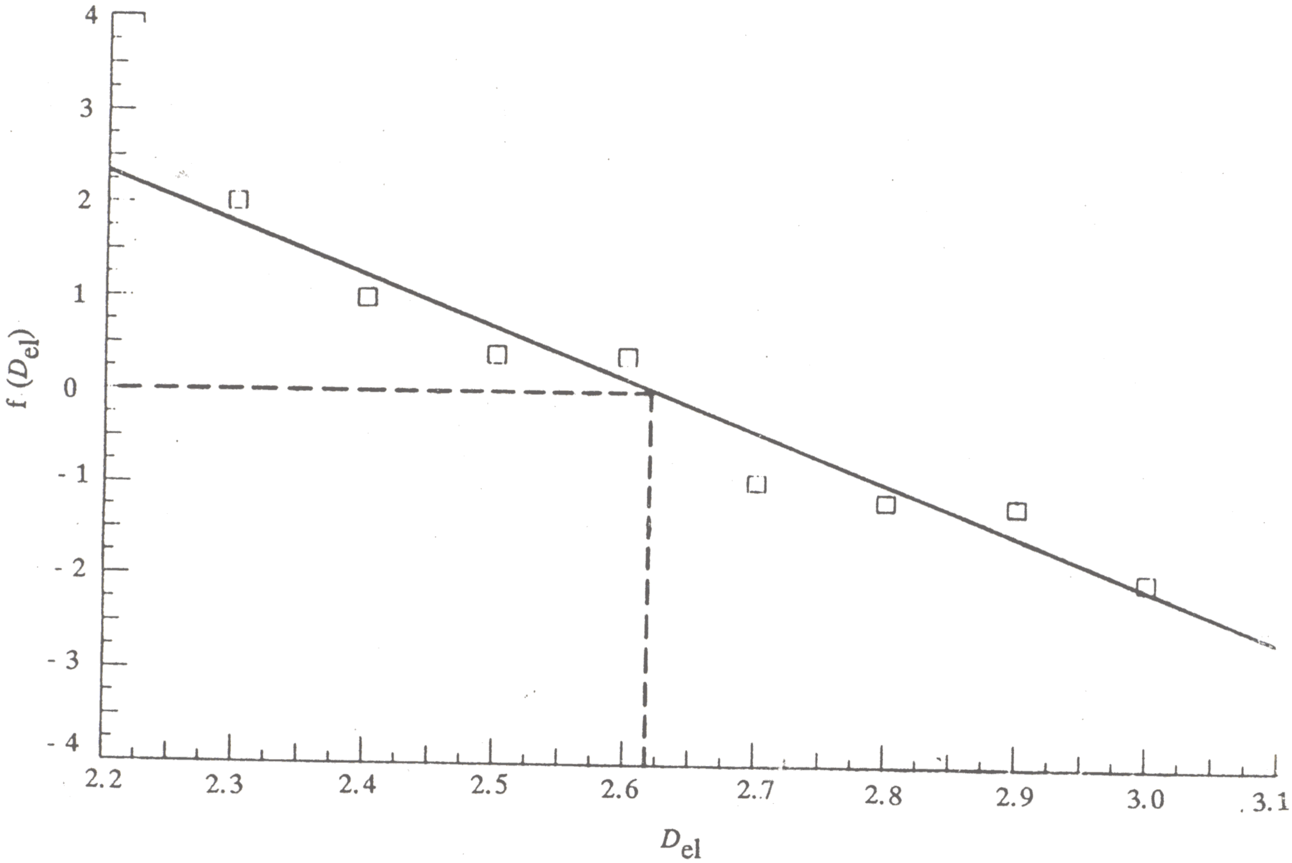

Fig.2. The function $f(D_{el})$ as it is defined in the text. It is based on eight values of the elliptical dimension of the box-counting space, D_{el} and ten reflectivity thresholds. Considering (Schertzer and Lovejoy, 1987) that $CD_{el}(T) = C_2(T)D_{el}/d_{el}$, it can be shown that $f(D_{el}) = (D_{el}/d_{el}-1)\Sigma C_2(T_i)$ which indicates that $f(D_{el})$ is linear. From this graph it is estimated that d_{el}= 2.62

(1) $\quad f(D_{el}) = \sum [CD_{el}(T_i) - C_2(T_i)]$

where the sum is over the number of thresholds which in our case is ten and D_{el} is the elliptical dimension of the box-counting space. The quantity $C(T)$ denotes the co-dimension of the field as defined by the threshold T. By definition $C_d(T) = d - D_d(T)$ where d is the dimension of the embedding Euclidean space and $D_d(T)$ is the fractal dimension of the field as a function of the threshold. Having previously defined the elliptical dimension, the philosophy behind estimating it via equation (1) is as follows: For isotropic fields

$$C_3(T) = 3 - D_3(T).$$

Also for isotropic fields one should expect that

$$D_3(T) = 1 + D_2(T)$$

(that is, taking cross-sections reduces the dimension by one). Thus, it follows that

$$C_3(T) = 2 - D_2(T) = C_2(T)$$

(the co-dimensions are conserved). It can be shown that conservation of co-dimensions extends to any subspace of the space the process is embedded in. For stratified data one obtains that

$$CD_{el}(T) = C_2(T)$$

(Lovejoy et al., 1987, Schertzer and Lovejoy 1987) as long as the data are analyzed with the correct stratification of the field hence with the correct elliptical dimension, d_{el}, of the field. We, therefore, seek the zero of the function in Eq.1. Eq.1. the quantities $CD_{el}(T_i)$ and $C_2(T_i)$ are of our data when employng box-counting, instead of Cartesian grids we used sectorial (pie-shaped) grids. The angular and downrange grid sizes increased by factor of 2 in lunear scale. By analyzing data in the (r,θ) space instead of transforming to a Cartesian coordinate system (x,y) we eliminate all averaging and range-dependent effects such as beam spreading, and so forth.

An example of box-counting from our radar reflectivity data is shown in Fig.1. The straight lines of the graph indicate that scaling is followed over corresponding range of scales. Note how the slope decreases with increasing T. That indicates that as the threshold increases the fractal dimension of the corresponding rain field decreases.

Fig.2 shows the function $f(D_{el})$ for the (x,y,t) rain field. From this figure we find that d_{el} is equal to 2.62. This result together with the previous estimate (Lovejoy ey al., 1987) d_{el} = 2.22 for rain in the (x,y,z) space produces an estimate of

$$d_{el} = 2.84 \ (= 2.22 + 2.62 - 2.00)$$

for rain in the (x,y,z,t) space. We should note at this point that the radar averages in space and samples in time. In order to address the effect of having instantaneous values in time instead of averages we experimented with averaging the data in time. We found no differences in the results. Thus, we decided to consider the (x,y,t) rainfield at its highest possible resolution. Our results indicate that even for short time scales (<20 minutes) the rain field is strongly anistropic. If the field were isotropic (d_{el}=3.0 in the (x,y,z) or (x,y,t) space or d_{el}=4.0 in the (x,y,z,t) space) then the statistical properties along any direction in space would be the same as those in time and thus the space/time transformations would be scale independent. Such a result would correspond to a statistical version of the "frozen turbulence" hypothesis put forward by G.I Taylor (Taylor, 1938) which assumes that the flow pattern is "frozen" and simply advected past the point of observation at a fixed velocity.

The validity of Taylor's hypothesis in the atmosphere has received a great deal of attention. The use of the hypothesis has the great advantage of enabling a single probe to infer spatial structures from time series. Becausee of that noted theoretical hydrologists (Waymire and Gupta, 1981; Gupta and Waymire, 1987; Rodriquez-Iturbe et al. , 1984; Waymire et al. , 1984) have stressed the importance of the hypothesis in stochastic rainfall modeling. Our results suggest that the validity of Taylor's hypothesis in the atmosphere has to be thought via anisotropic scaling and thus via space-time transformations which involve a scale dependent velocity rather than a constant velocity. This is important not only because it still enables one to estimate spatial statistical properties from temporal properties, but also because it is required in both non-scaling stochastic models of rain (Waymire and Gupta, 1981; Gupta and Waymire, 1987; Rodriquez-Iturbe et al. , 1984) as well as in fractal (Lovejoy and Mandelbrot, 1984) or multifractal models (Schertzer and Lovejoy, 1987). Our results, which are obtained using a new approach, seem to disagree with previous results obtained using correlations and spectra in rain (Zawadski, 1973; Crane, 1990) which claim that the "frozen turbolence" hypothesis should hold for time scales less than half an hour. We hope that our report will stimulate further research in this very important topic and approach.

Conclusions

In this report we presented results that indicate that even for short time scales the field is anisotropic. Our results suggest that space/time transformations in the atmosphere (and thus the validity of Taylor's hypothesis) are scale depended. Our approach provides an excellent example of how concepts from fractal geometry may help us address one of the most important theoretical issues of geophysical fluid dynamics. The conclusions drawn present a new challenge for rainfall theory and modeling as well as to the validity of Taylor's hypothesis and its relevence to scaling.

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